Nonlinear bang-bang eigenproblems and optimization of resonances in layered cavities

Illya M. Karabash [a,b,c,*], Olga M. Logachova [d], Ievgen V. Verbytskyi [e]

[a] Institut für Mathematik, Universität zu Lübeck, Haus 64, 3. OG Ratzeburger Allee 160, D-23562 Lübeck, Germany

[b] Institute of Applied Mathematics and Mechanics of NAS of Ukraine, Dobrovolskogo st. 1, Slovyans'k 84100, Ukraine

[c] Donetsk National University, Department of Mathematical Analysis and Differential Equations, Faculty of Mathematics and Information Technology, 600-richya st. 21, Vinnytsia 21021, Ukraine

[d] Siberian State University of Geosystems and Technologies, Department of Higher Mathematics, Institute of Geodesy and Management, Plakhotnogo st. 10, Novosibirsk 630108, Russia

[e] National Technical University of Ukraine "Kyiv Polytechnic Institute", Department of Industrial Electronics, Faculty of Electronics, Politekhnichna st. 16, block 12, Kyiv 03056, Ukraine

[*] Corresponding author. E-mail: i.m.karabash@gmail.com

Abstract

Quasi-normal-eigenvalue optimization is studied under constraints $b_1(x) \leq B(x) \leq b_2(x)$ on structure functions B of 2-side open optical or mechanical resonators. We prove existence of various optimizers and provide an example when different structures generate the same optimal quasi-(normal-)eigenvalue. To show that quasi-eigenvalues locally optimal in various senses are in the spectrum Σ^{nl} of the bang-bang eigenproblem $y'' = -\omega^2 y[b_1 + (b_2 - b_1)\chi_{\mathbb{C}_+}(y^2)]$, where $\chi_{\mathbb{C}_+}(\cdot)$ is the indicator function of $\mathbb{C}_+$, we obtain a variational characterization of Σ^{nl} in terms of quasi-eigenvalue perturbations. To address the minimization of the decay rate $|\operatorname{Im}\omega|$, we study the bang-bang equation and explain how it excludes an unknown optimal B from the optimization process. Computing one of minimal decay structures for 1-side open settings, we show that it resembles gradually size-modulated 1-D stack cavities introduced recently in Optical Engineering. In 2-side open symmetric settings, our example has an additional centered defect. Nonexistence of global decay rate minimizers is discussed.

Contents

1   Introduction

The paper is devoted to the analytical and numerical study of two connected questions: nonlinear eigenproblems with a bang-bang term and optimization of quasi-(normal-)eigenvalues in optical and mechanical resonators.

A leaky cavity (or resonator) is a region of space within which the electromagnetic (or acoustic) field is well confined, but not completely confined. Cavities supporting eigenmodes with high quality factor (high-Q) are needed for a number of applications including Cavity Quantum Electrodynamics, Optical Engineering, and Microscopy (see e.g. [45, 34, 35, 40]). The recent progress in fabrication of small size optical resonators [1, 26, 29] attracted considerable interest to numerical [19, 15, 3, 11, 29, 31] and analytical [21, 20, 23] aspects of resonance optimization.

Optical quasi-eigenvalues (or resonances) can be mathematically defined as eigenvalues ω corresponding to outgoing solutions of the time-harmonic Maxwell equation [41, 29]. Because of the leakage, resonant eigenoscillations are expected to decay exponentially in time. The modulus of the imaginary part $|\operatorname{Im}\omega|$ of a quasi-eigenvalue corresponds to *the decay rate* of the standing wave, the real part $\operatorname{Re}\omega$ to *the frequency of oscillations*. In Engineering studies, the confinement of energy for a particular resonant eigenmode is measured by *the quality factor* $Q(\omega) = \frac{\operatorname{Re}\omega}{-2\operatorname{Im}\omega}$.

Since light is essentially difficult to localize it is hard to realize small-sized optical cavities with strong light confinement. One of the reasons of active involvement of Numerics in this field is that the initial progress with fabrication of high-Q cavities based on 2-D photonic crystals was achieved with the help of computer simulations of eigenmodes for perspective versions of design [1]. The interest to multilayer structures with 1-D geometry has soon returned [33, 32, 26, 3, 31], partially because of cheaper fabrication process, partially because computations in this case are simpler. For idealized 1-D cavities with infinite layers and under the assumption of normally passing electromagnetic (EM) waves, the Maxwell system can be reduced to a 1+1 wave equation of nonhomogeneous string $B(x)\partial_t^2 u(x,t) = \partial_x^2 u(x,t)$ (see e.g. [43, 40]). The analogy with a string is widely used in Optical Engineering [27, 30] (and possibly goes back to Lord Rayleigh [17, Chapter 4]).

Quasi-eigenvalues associated with $B(x)\partial_t^2 u(x,t) = \partial_x^2 u(x,t)$ can be defined via the eigenproblem consisting of the time-harmonic equation

$$y''(x) \;\; = \;\; -\omega^2 B(x)y(x) \qquad \text{a.e. on } a_1 < x < a_2, \tag{1.1}$$

equipped with radiation boundary conditions. In this 1-D case, the radiation conditions take the ω-dependent form of local conditions at the endpoints

$$\frac{y'(a_1)}{-i\omega} \;\; = \;\; \nu_1 y(a_1), \tag{1.2}$$

$$\frac{y'(a_2)}{i\omega\nu_2} \;\; = \;\; y(a_2) \qquad \text{(see Section 2 for details).} \tag{1.3}$$

The following properties of the *set $\Sigma(B)$ of quasi-eigenvalues associated with the coefficient $B(\cdot)$* are important for the present paper: $\Sigma(B)$ is a subset of the lower complex half-plane $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ and is symmetric with respect to (w.r.t.) the imaginary axis $i\mathbb{R}$, quasi-eigenvalues are isolated and of finite algebraic multiplicity, ∞ is their only possible accumulation point (see Section 3 and, for one-side open cases, [24, 25, 7, 12, 36]).

From mathematical point of view, estimates on resonances associated with various wave equations have being studied in Mathematical Physics at least since 1970s [28] and is still an active area of research (see e.g. [2] and references therein). Optimization of resonances may be seen as an attempt to obtain sharp estimates of this kind. This point of view and the study of resonances associated with random Schrödinger operators were initial sources of the interest to the problem [13, 14]. It

seems that, up to now, sharp estimates have been obtained only in low frequency regions for a few 1-D models involving total mass type constraints [20, 23] (on a somewhat different problem involving damping, see [6] and the discussion in [5]). Among other variational problems for eigenvalues, the optimization of resonances can be classified as the nonselfadjoint spectral optimization. It contains essentially new effects and difficulties in comparison with selfadjoint cases (see [7, 5, 4, 21, 23]).

Since quasi-eigenvalues ω are complex numbers, one of the ways to formulate related variational problems is to interpret ω as an $\mathbb{R}^2$-vector and to take the point of view of the optimization theory for vector-valued cost functions (Pareto optimization) [23]. The approach of [23] is a development of that of [21], where the engineering problem of the high-Q design for one-side open multilayer cavities was considered analytically. To provide a simple and simultaneously rigorous formulation, the papers [20, 21] restate the problem of high-Q design in terms of the minimization of the decay rate $|\operatorname{Im}\omega|$ for quasi-eigenvalues with a fixed frequency $\operatorname{Re}\omega = \alpha$. Then the set of all *quasi-eigenvalues with minimal possible decay for their particular frequencies* forms a *Pareto optimal frontier* (see Fig. 1 in Section 6 and [23, Fig. 1]).

The quasi-eigenvalues are supposed to be generated by cavities with structures belonging to a certain admissible family. In the present paper, the resonator's *structure* is described by an L^1-function $B(x)$ on $[a_1, a_2]$ $(-\infty < a_1 < a_2 < +\infty)$, which represents spatially varying dielectric permittivities in the case of optical cavity, or the linear density in Mechanics settings. Boundary conditions at the endpoints $a_{1,2}$, which describe the leakage of energy due to radiation or damping, include two parameters $\nu_1, \nu_2 \in [0, +\infty]$. These parameters are assumed to be fixed, and so, they do not participate in the optimization process. The admissible family of the structures $\mathbb{A}$ (over which the optimization is performed) is defined by the functional side constraints $b_1(x) \le B(x) \le b_2(x)$ a.e. on (a_1, a_2). This allows us to consider an important for applications situation when the resonator itself is only a part of a more complex device [32, 29]. That is, sometimes only some parts of the system are suitable for modifications to achieve better resonant properties. The part of the device that is not subjected to optimization corresponds to the intervals where $b_1(x) = b_2(x)$.

The connection between quasi-eigenvalues of minimal decay and nonlinear eigenproblems was established recently in [21, 22] for the case of 1-D models of one-side open resonators and constant side constraints b_1 and b_2. It was proved that the set of Pareto optimal quasi-eigenvalues of minimal decay belongs to the spectrum associated with the equation $y''(x) = -\omega^2 y(x) \left[b_1 + [b_2 - b_1]\chi_{\mathbb{C}_+}(y^2(x)) \right]$.

The nonlinearity contains the function of bang-bang type $\mathcal{B}(y)(x) = \left[b_1 + (b_2 - b_1)\chi_{\mathbb{C}_+}(y^2) \right]$ taken of the square of complex-valued solution y. Here and below $\chi_{\mathbb{C}_+}(\cdot)$ is the indicator (characteristic) function of the open upper complex half-plane $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$, i.e.,

$$\chi_{\mathbb{C}_+}(z) = 1 \text{ if } z \in \mathbb{C}_+, \text{ and } \chi_{\mathbb{C}_+}(z) = 0 \text{ if } z \in \overline{\mathbb{C}_-} = \{z : \operatorname{Im} z \le 0\}.$$

So $\mathcal{B}(y)(\cdot)$ equals $b_1(\cdot)$ on the x-intervals where $\operatorname{Im} y^2(x) \le 0$, and $\mathcal{B}(y)(\cdot) = b_2(\cdot)$ for x-intervals where $\operatorname{Im} y^2(x) > 0$. This resembles to some extend bang-bang equations arising in the optimal control theory when the control is restricted by a lower and an upper bound. However, the interpretation of the above equation from the point of view of control is not found yet.

Since the nonlinear equation derived in [21] excludes infinite-dimensional unknown structure B, it may essentially simplify computation of optimal resonances in comparison with step-by-step adjustment procedures for B based on computation of electromagnetic field on each step.

Development and analytic justification of a numerical procedure based on the bang-bang eigenproblem is one of the goals of our research. However, the implementation of this new computational approach requires better understanding of the shapes of the nonlinear spectrum and of the Pareto optimal set, as well as their projections to the real line (see Section 6.2). Our method also requires the study of properties of solutions of the bang-bang equation (1.4), which are considered in Section 10.3 and Appendix A.4.

We show that not only quasi-eigenvalues of minimal decay, but also quasi-eigenvalues locally extremal in a much wider sense belong to the set Σ^{nl} of eigenvalues associated with the nonlinear equation

$$y''(x) = -\omega^2 y \left[b_1(x) + [b_2(x) - b_1(x)] \chi_{\mathbb{C}_+}(y^2(x)) \right], \quad a_1 < x < a_2 \qquad (1.4)$$

(which includes the functional side constraints $b_{1,2}(\cdot)$ and is supposed to be equipped with the radiation boundary conditions). In particular, we show that *nonlinear spectrum* Σ^{nl} contains the nonimaginary part $\mathrm{bd}\,\Sigma[\mathbb{A}] \setminus i\mathbb{R}$ of the boundary $\mathrm{bd}\,\Sigma[\mathbb{A}]$ of the *set of admissible quasi-eigenvalues*

$$\Sigma[\mathbb{A}] := \bigcup_{B \in \mathbb{A}} \Sigma(B) \qquad \text{(see Sections 2 and 9)}.$$

To achieve this, we provide a variational characterization of the part of Σ^{nl} lying in $\mathbb{C} \setminus i\mathbb{R}$ (Theorem 5.1). We also show that $\mathrm{bd}\,\Sigma[\mathbb{A}]$ and so Σ^{nl} are, at least in some cases, essentially 'greater' than the set of quasi-eigenvalues of minimal decay (see Sections 8 and 11 and Example 10.3).

The quasi-eigenvalue of minimal decay for a particular frequency α can be found as the closest to $\mathbb{R}$ point of the intersection of the nonlinear spectrum with the line $\alpha + i\mathbb{R}$ (see Section 10). Then, corresponding optimal structures B can also be found from the bang-bang eigenproblem. In the end of the paper, we support our study by a numerical experiment (see Fig. 2 (a) and Tables 1-2 in Section 11).

To prove rigorously that points of the nonlinear spectrum are not necessarily quasi-eigenvalues of minimal decay, the case of small dielectric contrast is considered in Section 8. In Optics literature the study of small contrast is often serves as a base of intuitive understanding of qualitative effects (see e.g. [17, Chapter 4]). In the optimization settings, small contrast means that the constraints b_1 and b_2 are close to each other. To measure the corresponding distance we use the L^1-norm $\|b_2 - b_1\|_1$ since this choice is flexible enough to cover both the small changes in permittivity function (when $\|b_2 - b_1\|_\infty$ is small) and fluctuations in the widths and positions of layers. This gives a connection with models involving uncertainties or random deviations in the resonator's structure $B(x)$. Indeed, for stochastic models, all possible positions of random resonances (the supports of probabilistic distributions) can be localized by the set of admissible resonances $\Sigma[\mathbb{A}]$ of a properly posed optimization problem. In practical situations uncertainties or fluctuations in the cavity's structure may model unintended deviations in fabrication process [46], incorporation of biomolecular thin films in X-ray multi-layer cavities [35], and various Physics effects affecting EM properties of materials [16].

Pure imaginary quasi-eigenvalues (i.e., the case $\omega \in i\mathbb{R}$) correspond to the over-damping and critical damping effects (in Mechanics settings). For them, optimization partially resembles that for selfadjoint problems [6, 7, 5, 21]. The pure imaginary case is considered separately in Section 7. It provides, as a by-product, an example of two different optimal structures associated with one quasi-eigenvalue of minimal decay. This partially answers the uniqueness of optimizer question discussed in [14, 20, 23].

The application of steepest ascent numerical procedures to the search of quasi-eigenvalues with locally minimal decay rate (or locally maximal quality factor) attracted a considerable attention [15, 19, 40]. It seems that Corollary 7.3 (i) and Theorem 8.1 provide *first rigorous proofs of existence of such local minimizers.* (The proof of Theorem 8.1 can be easily transformed to consider local maximizers for the Q-factor). *The existence of global minimizer for decay rate* (over all $B \in \mathbb{A}$ and $\omega \in \mathbb{C}_-$ such that $\omega \in \Sigma(B)$) *is questionable.* Our opinion is that it is difficult to expect such existence in most of practical situations, see the discussion in Section 12.2.

The discussion section (Section 12) also compares the results of the numerical experiment of Section 11 with optimal design suggestions of engineering and numerical optimization papers. Our conclusion is that the gradually size-modulated 1-D stack designs of [33, 26, 3] are reasonable, but

for symmetric resonators, it is possible that *an additional defect in the center may improve the confinement of energy.*

The results of the paper were partially reported at the International Congress of Mathematicians (Seoul ICM 2014) and in several seminar and workshop talks at the University of Lübeck, the Institute of Mathematics (at Kiev), B. Verkin Institute for Low Temperature Physics and Engineering, and Pidstryhach Institute for Applied Problems of Mechanics and Mathematics.

Notation. By $\lceil x \rceil$ and $\lfloor x \rfloor$ the ceiling and floor functions are denoted, i.e., the smallest integer not less than x and the greatest integer not greater than x, respectively.

The following sets of real and complex numbers are used: open half-lines $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$, and open discs $\mathbb{D}_\epsilon(\zeta) := \{z \in \mathbb{C} : |z - \zeta| < \epsilon\}$ with the center at ζ and radius ϵ. The Lebesgue measure of a set $S \subset \mathbb{R}$ is denoted $\operatorname{meas} S$.

$L^p_{\mathbb{C}(\mathbb{R})}(a_1, a_2)$ are the Lebesgue spaces of complex- (resp., real-) valued functions and

$$W^{k,p}[a_1, a_2] := \{y \in L^p_\mathbb{C}(a_1, a_2) \ : \ \partial^j_x y \in L^p_\mathbb{C}(a_1, a_2), \ \ 1 \le j \le k\}$$

are Sobolev spaces. The corresponding standard norms are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}}$. The space of continuous complex valued functions with the uniform norm is denoted by $C[a_1, a_2]$.

Under the support $\operatorname{supp} B$ of $B \in L^1(a_1, a_2)$ we understood the topological support of the absolutely continuous measure $|B(x)|\mathrm{d}x$ on $[a_1, a_2]$, i.e., $\operatorname{supp} B$ is the smallest closed set S such that $B(x) = 0$ a.e. on $[a_1, a_2] \setminus S$.

For basic definitions of convex analysis we refer to [39]. Let S be a subset of a linear space U over $\mathbb{C}$ (including the case $U = \mathbb{C}$). For $u_0 \in U$ and $z \in \mathbb{C}$, $zS + u_0 := \{zu + u_0 \ : \ u \in S\}$. The convex cone generated by S (the set of all nonnegative linear combinations of elements of S) is denoted by $\operatorname{cone} S$. Open balls in a normed space U are denoted by

$$\mathbb{B}_\epsilon(u_0) = \mathbb{B}_\epsilon(u_0; U) := \{u \in U \ : \ \|u - u_0\|_U < \epsilon\}.$$

The closure of a set S (in the norm topology) is denoted by $\overline{S}$, the boundary of S by $\operatorname{bd} S$. For a function f defined on S, $f[S]$ is the image of S. By $\partial_x f$, $\partial_z f$, etc., we denote (ordinary or partial) derivatives w.r.t. x, z, etc. The function f_+ is 'the positive part' of f, i.e., $f_+(x) := \begin{cases} f(x) & \text{, if } f(x) \ge 0 \\ 0 & \text{, if } f(x) < 0 \end{cases}$; χ_S is an indicator function of a set S, i.e., $\chi_S(x) = 1$ when $x \in S$, and $\chi_S(x) = 0$ when $x \notin S$.

2 Basic settings, quasi-eigenvalues and admissible sets

The boundary conditions (1.2) and (1.3) at $a_{1,2} \in \mathbb{R}$ involve the (extended) constants $\nu_{1,2}$, which throughout the paper are assumed to satisfy

$$\nu_1 \in [0, +\infty), \qquad \nu_2 \in (0, +\infty], \qquad |\nu_1| + |1/\nu_2| \ne 0, \quad \text{and } \nu_1 \le \nu_2. \tag{2.1}$$

When damping coefficients $\nu_{1,2}$ are fixed, we can define the *set of quasi-eigenvalues* $\Sigma(B)$ associated with the structure B as the set of $\omega \in \mathbb{C} \setminus \{0\}$ such that eigenproblem (1.1), (1.2), (1.3) has a nontrivial solution $y \in W^{2,1}[a_1, a_2]$ (i.e., a solution that is not identically zero). This solution y is called a (quasi-normal) *mode*. Several other names for ω are used, sometimes in slightly different settings: scattering poles [28], dissipation frequencies [24, 25], resonances [10, 19], quasi-normal levels (in the Quantum Physics literature).

The function $B \in L^1_\mathbb{R}(a_1, a_2)$ in (1.1) describes the *structure* of a non-homogeneous medium varying in x-direction. In the case of optical cavity, $B(x_0)$ corresponds to permittivity of the transparent dielectric material of the layer with the x-coordinate equal to x_0 (if the light speed in vacuum is

normalized to be 1 or if one works with the complex wave number instead of the complex angular frequency ω). In Mechanics models involving the equation of nonhomogeneous string, the function B is the varying linear density of the string (if the tension equals 1). So, in these models, B is supposed to be a.e. nonnegative, and we will keep this assumption in the context of optimization problems. However, in Section 3, the definition of quasi-eigenvalues will be extended to complex-valued coefficient B for the needs of perturbation theory of Section 4.

When ν_j is in $(0, +\infty)$, the corresponding boundary condition is dissipative and describes either linear damping at a_j or the radiation of waves into outer regions (see the examples in Section 6.2 for details). Our settings for a 1-D resonator is a slightly generalized version of [32, 42]. We allow B to be equal to 0 on certain sets (the massless string approximation). We also include the cases when one of the boundary conditions is conservative (and so the resonator is only one-side open). In particular, when $\nu_2 = +\infty$, we suppose that $1/\nu_2 = 0$, and then (1.3) turns into the Dirichlet condition $y(a_2) = 0$. Note that the assumption $\nu_1 \leq \nu_2$ does not restrict the generality since, in the case $\nu_1 > \nu_2$, one can use the change of variable $\widetilde{x} = -x$.

While most of mathematical studies were devoted to one-side open or symmetric resonators (mathematically, these two cases are almost equivalent, see Example 6.4), contemporary Optics applications often involve the two-side open case [33, 42, 26, 3, 31]. In the present paper, we consider one- and two-side open resonators in a unified way. Besides the importance for engineering applications, the study of two-side open resonators brings new mathematical effects. In particular, the spatial phase $\arg y(x)$ of the complex-valued mode y is not monotone in contrast to one-side open case. This effect, and the fact that the notion of switch points (see [21]) looses its natural sense when the constraint functions $b_{1,2}$ are allowed to be equal on a set of positive measure, lead in Section 5 to more detailed study of rotational properties of resonant modes and to more essential use of Convex Analysis in comparison with the technique of [21].

We consider optimization over the following family of *admissible structures*

$$\mathbb{A} := \{B(x) \in L^1_{\mathbb{R}}(a_1, a_2) \ : \ b_1(x) \leq B(x) \leq b_2(x) \text{ a.e. on } (a_1, a_2)\}, \tag{2.2}$$

where $b_{1,2}$ are certain Lebesgue integrable functions defined on (a_1, a_2) such that

$$0 \leq b_1(x) \leq b_2(x) \text{ on } (a_1, a_2) \qquad \text{and} \tag{2.3}$$

$$\text{meas } E > 0, \text{ where } E = \{x \in (a_1, a_2) \ : \ b_1(x) < b_2(x)\}. \tag{2.4}$$

A complex number ω is called *an admissible quasi-eigenvalue* if it belongs to the set $\Sigma[\mathbb{A}] := \bigcup_{B \in \mathbb{A}} \Sigma(B)$. Keeping in mind applications, we will pay main attention to the problem of minimization of decay rate of an individual resonance ω generated by a certain $B \in \mathbb{A}$. However, from mathematical point of view it is more convenient to consider ω itself as a cost function with the values in $\mathbb{R}^2$ and to study extremal in various senses quasi-eigenvalues . One of such sets is the boundary $\operatorname{bd}\Sigma[\mathbb{A}]$ of the *set of admissible quasi-eigenvalues* $\Sigma[\mathbb{A}]$.

Actually, ω can not be considered directly as a functional of B even locally due to multiplicity and splitting issues (see Sections 3.2, Proposition 4.4, and the discussion in [23]). However, a rigorous approach to $\omega(B)$ can be given via the set-valued map $B \mapsto \Sigma(B)$. Local extrema of such set-valued maps were introduced in [23] and will be the key tool in Section 9. In such generalized settings, $\operatorname{bd}\Sigma[\mathbb{A}]$ plays a role of a set of generalized Pareto extremizers for the map $B \mapsto \Sigma(B)$ over $\mathbb{A}$.

3 Properties of resonances and related maps

3.1 Characteristic determinant F and definition of multiplicity

In this section, an equivalent (but more convenient from the point of view of perturbation theory) definition of quasi-eigenvalues is given under more general assumption that $B \in L^1_{\mathbb{C}}(a_1, a_2)$.

Denote by $\varphi(x) = \varphi(x, z; B)$, $\psi(x) = \psi(x, z; B)$, and $\theta(x) = \theta(x, z; B)$ the solutions to $\partial_x^2 y(x) = -z^2 B(x) y(x)$ satisfying

$$\varphi(a_1, z; B) = \partial_x \psi(a_1, z; B) = 1, \qquad \partial_x \varphi(a_1, z; B) = \psi(a_1, z; B) = 0,$$
$$\theta(a_1, z; B) = 1, \qquad \partial_x \theta(a_1, z; B) = -\mathrm{i} z \nu_1. \tag{3.1}$$

Obviously, $\theta(x) = \varphi(x) - \mathrm{i} z \nu_1 \psi(x)$ and $\theta(x)$ is a unique solution to the integral equation

$$y(x) \;=\; 1 - \mathrm{i} z \nu_1 (x - a_1) - z^2 \int_{a_1}^x (x - s)\, y(s)\, B(s) \mathrm{d}s, \quad a_1 \le x \le a_2. \tag{3.2}$$

Recall that the set of quasi-eigenvalues corresponding to a structure B (in short, quasi-eigenvalues of B) is denoted by $\Sigma(B)$. The following lemma is the integral reformulation of the quasi-eigenvalue problem.

Lemma 3.1. *A number $\omega \in \mathbb{C}$ belongs to $\Sigma(B)$ if and only if there exists nontrivial $y(x) \in C[0, \ell]$ satisfying equation (3.2) with $z = \omega$ and the equality*

$$y(a_2) + \frac{\nu_1}{\nu_2} - \frac{\mathrm{i}\omega}{\nu_2} \int_{a_1}^{a_2} y(s)\, B(s) \mathrm{d}s = 0. \tag{3.3}$$

Note that in the integral settings there is no need to exclude separately the case $\omega = 0$. If $\omega = 0$, then it is easy to see that the problem consisting of (3.2) with $z = \omega$ and (3.3) has no nontrivial solutions.

Clearly, $\Sigma(B)$ is the set of zeroes of the function $F(\cdot; B)$, where

$$F(z) = F(z; B) := \theta(a_2, z; B) + \frac{\nu_1}{\nu_2} - \frac{\mathrm{i}z}{\nu_2} \int_{a_1}^{a_2} \theta(s, z; B)\, B(s) \mathrm{d}s. \tag{3.4}$$

It is easy to see that for $z \ne 0$,

$$F(z; B) = \theta(a_2, z; B) + \frac{\mathrm{i} \partial_x \theta(a_2, z; B)}{z \nu_2}. \tag{3.5}$$

When $\nu_2 = \infty$, these formulas turns into $F(z; B) = \theta(a_2, z; B)$.

We say that a map $G : U_1 \to U_2$ between normed spaces $U_{1,2}$ is *bounded-to-bounded* if the set $G[S]$ in U_2 is bounded for any bounded S in U_1.

Lemma 3.2. (*i*) *The functional $F(z; B)$ is analytic on the Banach space $\mathbb{C} \times L^1(a_1, a_2)$.*

(*ii*) *Let $1 \le p \le \infty$. The mappings $(z, B) \mapsto \varphi(\cdot, z; B)$, $(z, B) \mapsto \psi(\cdot, z; B)$, and $(z, B) \mapsto \theta(\cdot, z; B)$ are bounded-to-bounded and analytic from $\mathbb{C} \times L_{\mathbb{C}}^p(a_1, a_2)$ to $W^{2,p}[a_1, a_2]$.*

The proof is given in Appendix A.1 (concerning analytic mappings on Banach spaces, see [37]).

It is obvious that all modes y corresponding to $\omega \in \Sigma(B)$ are equal to $\theta(\cdot, \omega; B)$ up to a multiplication by a constant. So *the geometric multiplicity* of any quasi-eigenvalue equals 1. In the following, *the multiplicity* of a quasi-eigenvalue means its *algebraic multiplicity*.

Definition 3.1. *The multiplicity* of a quasi-eigenvalue of B is its multiplicity as a zero of the entire function $F(\cdot; B)$. A quasi-eigenvalue is called *simple* if its multiplicity is 1. The set of non-simple quasi-eigenvalues is denoted by $\Sigma_{\mathrm{mult}}(B)$ (non-simple quasi-eigenvalues are often called multiple, or degenerate).

This is essentially the classical M.V. Keldysh definition of multiplicity for eigenvalue problems with an eigen-parameter in boundary conditions (note that F coincides with the characteristic determinant of (1.1)-(1.3) up to a constant), see [24, 25, 7, 36, 21, 23] and references therein.

Since $F(0; B) = 1 + \frac{\nu_1}{\nu_2} > 0$, each quasi-eigenvalue has a finite multiplicity and the set $\Sigma(B)$ consists of isolated points, which can accumulate only to ∞. Note that $\Sigma(B)$ may be empty, see Proposition 3.4.

3.2 Non-simple resonances, the case of nonnegative B, and examples

There exist triples (ν_1, ν_2, B), *consisting of numbers* $\nu_{1,2}$ satisfying the assumption (2.1) and a non-negative function B, *that generate non-simple quasi-eigenvalues.* This follows from [12] and [36, Theorem 4.1] (with $\nu_2 = +\infty$). For a slightly different class of quasi-eigenvalue problems, existence of degenerate quasi-eigenvalues was proved in [25, Theorem 3.1] (English translation of this theorem can be found in [20], see also [24]), and examples were given in [30, 20].

Lemma 3.3. *Let* $B(x) \geq 0$ *a.e. and* $\nu_{1,2}$ *satisfy (2.1). Then:*

(i) $\Sigma(B) \subset \mathbb{C}_-$,

(ii) $\Sigma(B)$ *is symmetric w.r.t. the imaginary axis* $\mathrm{i}\mathbb{R}$, *moreover, the multiplicities of symmetric quasi-eigenvalues coincide.*

Proof. **(i)** Let $\omega \in \Sigma(B)$ and y be an associated eigenfunction. The energy of the eigen-oscillation $u(x, t) = e^{-\mathrm{i}\omega t} y(x)$ at the time t is

$$\int_{a_1}^{a_2} |\partial_x u|^2 + B|\partial_t u|^2 \mathrm{d}x = e^{-2\beta t} \, \mathrm{En}_y, \quad \text{where} \quad \mathrm{En}_y := \int_{a_1}^{a_2} |y'|^2 + B \, |\omega|^2 \, |y|^2 \mathrm{d}x.$$

Since y is nontrivial, we see that $\mathrm{En}_y > 0$. (Otherwise, $y' \equiv 0$ and, in turn, $y \equiv 0$).

Recalling that $\omega \neq 0$ and taking the real part in the identity

$$-\mathrm{i}\omega \, \mathrm{En}_y = -\mathrm{i}\omega \int_{a_1}^{a_2} |y'|^2 \mathrm{d}x + \mathrm{i}\overline{\omega} \int_{a_1}^{a_2} (-\omega^2) B \, y \, \overline{y} \, \mathrm{d}x =$$

$$= -2\mathrm{i} \, \mathrm{Re}\, \omega \int_{a_1}^{a_2} |y'|^2 \mathrm{d}x - |\omega|^2 \left(\nu_1 |y(a_1)|^2 + \frac{|y'(a_2)|^2}{|\omega|^2 \nu_2} \right),$$

one gets

$$\mathrm{Im}\, \omega = -\frac{|\omega|^2}{\mathrm{En}_y} \left(\nu_1 |y(a_1)|^2 + \frac{|y'(a_2)|^2}{|\omega|^2 \nu_2} \right) < 0.$$

Note that inequality $\nu_1 |y(a_1)|^2 + \frac{|y'(a_2)|^2}{|\omega|^2 \nu_2} > 0$ follows from (2.1) and $y \not\equiv 0$.

(ii) follows from identities

$$\theta(x, \mathrm{i}\zeta; B) = \overline{\theta(x, \mathrm{i}\overline{\zeta}; B)} \quad \text{and} \quad F(\mathrm{i}\zeta; B) = \overline{F(\mathrm{i}\overline{\zeta})}. \tag{3.6}$$

$\square$

When the medium of a resonator is homogeneous, quasi-eigenvalues can be calculated explicitly.

Proposition 3.4. *Let* $B(x) \equiv b$, *where* $b \geq 0$ *is a constant.*

(i) *If* $b^{1/2} = \nu_1$ *or* $b^{1/2} = \nu_2$, *then* $\Sigma(b) = \varnothing$.

(ii) *If* $b > 0$, $b^{1/2} \neq \nu_1$, *and* $b^{1/2} \neq \nu_2$, *then* $\Sigma(b) = \{\omega_n(b)\}_{n=-\infty}^{+\infty}$ *with*

$$\omega_n(b) = -\frac{\mathrm{i}}{2b^{1/2}(a_2 - a_1)} \ln \left| \frac{1 + K_1}{1 - K_1} \right| + \frac{\pi}{b^{1/2}(a_2 - a_1)} \times \left\{ \begin{array}{ll} n, & \text{if } b^{1/2} \notin [\nu_1, \nu_2] \\ (n + 1/2), & \text{if } b^{1/2} \in (\nu_1, \nu_2) \end{array} \right., \tag{3.7}$$

where $K_1 := \frac{1 + \nu_1/\nu_2}{\frac{\nu_1}{b^{1/2}} + \frac{b^{1/2}}{\nu_2}} > 0$ *and* $n \in \mathbb{Z}$.

(iii) *If* $b = 0 \neq \nu_1$, *then* $\Sigma(b) = \{-\mathrm{i}\frac{1/\nu_1 + 1/\nu_2}{a_2 - a_1}\}$.

In the cases (ii)-(iii) all quasi-eigenvalues are simple.

Proof. Consider the case $b > 0$. The equation $F(z) = 0$ can be transformed into $\tan([a_2 - a_1]b^{1/2}z) + iK_1 = 0$ and we know that the roots ω are in $\mathbb{C}_-$. Taking ∂_z one can see that the roots are simple. We have $K_1 = 1$ exactly when $b^{1/2} \in \{\nu_1, \nu_2\}$. In this case $\Sigma(b) = \varnothing$. In the case $K_1 \neq 1$, the real part of $\mathrm{Arctan}(-\mathrm{i}K_1)$ is either πn or $\pi(n + 1/2)$ depending on the sign of $1 - K_1$. This leads to (3.7). Note that

$$1 > K_1 \;\Leftrightarrow\; b^{1/2} \notin [\nu_1, \nu_2], \quad 1 < K_1 \;\Leftrightarrow\; b^{1/2} \in (\nu_1, \nu_2), \quad \text{and } K_1 = 1 \;\Leftrightarrow\; b^{1/2} \in \{\nu_1, \nu_2\}. \quad (3.8)$$

The case $b = 0$ can be treated by straightforward computations. $\qquad\square$

In the particular cases of one-side open resonators (i.e., when $\nu_1 = 0$ or $\nu_2 = +\infty$) this example is well known [43, 7].

4 Perturbations and weak continuity of resonances

Let U be a Banach space. For a functional $G(z; u)$ that maps $\mathbb{C} \times U$ to $\mathbb{C}$, we denote by

$$\frac{\partial G(z; u_0)}{\partial u}(u_1) := \lim_{\zeta \to 0} \frac{G(z; u_0 + \zeta u_1) - G(z; u_0)}{\zeta},$$

the directional derivative of $G(z; \cdot)$ along the vector $u_1 \in U$ at the point $u_0 \in U$. By $\frac{\partial G(z; u_0)}{\partial u}$ the corresponding *Fréchet derivative in u* is denoted. If the Fréchet derivative $\frac{\partial G(z; u_0)}{\partial u}$ exists, it is a linear functional on U and $\frac{\partial G(z; u_0)}{\partial u}[S]$ defines the image of a set $S \subset U$ under this functional.

Remark 4.1 (Global discontinuity of $B \mapsto \Sigma(B)$). Before the study of continuity properties of quasi-eigenvalues, let us note that *new quasi-eigenvalues can appear from ∞ with small variations of B.* To see this, it is enough to take the homogeneous structure B_0 equal to ν_1^2 or ν_2^2 and to consider $\Sigma(B)$ using (3.7) and taking $B \equiv B_0 + \varepsilon$ with small $\varepsilon \in \mathbb{R}$.

This case is very special because, when $B_0 = \nu_j^2$, the resonator is not separated from outer space by any reflecting barrier (see detailed explanations in [7]). We *do not know if the above effect of global in ω discontinuity of the set-valued map $B \mapsto \Sigma(B)$ can happen in a vicinity of $B_0 \not\equiv \nu_{1,2}^2$.*

4.1 Local weak continuity

Let us fix a countable family $\{f_n\}_{n=1}^\infty$ of continuous functions that is dense in $C[a_1, a_2]$. This family generates a metric $\rho_{\mathbb{M}}(\mathrm{d}M_1, \mathrm{d}M_2) := \sum_{n=1}^\infty \dfrac{|\int f_n(\mathrm{d}M_1 - \mathrm{d}M_2)|}{2^n(1 + |\int f_n(\mathrm{d}M_1 - \mathrm{d}M_2)|)}$ on the space $\mathbb{M}$ of complex Borel measures on $[a_1, a_2]$. The weak* topology on any closed ball in $\mathbb{M}$ coincides with the topology generated by the metric $\rho_{\mathbb{M}}$. We will use the metric

$$\rho(B_1, B_2) := \rho_{\mathbb{M}}(B_1 \mathrm{d}x, B_2 \mathrm{d}x) \qquad \text{on } L^1_{\mathbb{C}}(a_1, a_2),$$

where $B_j \, \mathrm{d}x$ are absolutely continuous measures corresponding to the functions $B_j \in L^1_{\mathbb{C}}$.

In this subsection, $\mathbb{B}_R$ is the closed ball $\mathbb{B}_R(0)$ in $L^1_{\mathbb{C}}(a_1, a_2)$ with radius $R > 0$ and centered in the origin.

Lemma 4.1. *Let $z \in \mathbb{C}$.*

(i) The functional $F(z; \cdot) : \mathbb{B}_R \to \mathbb{C}$ is continuous w.r.t. the metric ρ.

(ii) The map $B \mapsto \theta(\cdot, z; B)$ is continuous from the metric space $(\mathbb{B}_R, \rho)$ to the normed space $C[a_1, a_2]$.

Proof. **(i)** follows from **(ii)**. Let us prove **(ii)**. Suppose $B_*, B_j \in \mathbb{B}_R$ and $\rho(B_j, B_*) \to 0$ as $j \to \infty$. In particular, $\mathrm{w}^*\text{-}\lim B_j \mathrm{d}x = B_* \mathrm{d}x$. By Lemma 3.2 (ii),

the sequence $\theta_j(\cdot) := \theta(\cdot, z; B_j)$ is bounded in $W_{\mathbb{C}}^{2,1}$, and so is relatively compact in $C[a_1, a_2]$. (4.1)

Hence there exists a subsequence θ_{j_k} (strongly) convergent in $C[a_1, a_2]$ to a certain θ_*. Using $\mathrm{w}^*\text{-}\lim B_j \mathrm{d}x = B_* \mathrm{d}x$ and passing to the limit in (3.2) for every $x \in [a_1, a_2]$, one can show that $\theta_*(\cdot) = \theta(\cdot, z; B_*)$.

Assume that θ_j does not converge to θ_*. Then (4.1) imply that there exists a subsequence θ_{n_k} convergent to $\theta_{**} \neq \theta_*$. Passing to the limit in (3.2) again, we see that $\theta_{**}(\cdot) = \theta(\cdot, z; B_*) = \theta_*(\cdot)$, a contradiction. Thus, $\theta(\cdot, z; B_j) \to \theta(\cdot, z; B_*)$ as $j \to \infty$. $\square$

The *total multiplicity of quasi-eigenvalues of B in a set* $\mathcal{D} \subset \mathbb{C}$ is the sum of multiplicities of all $\omega \in \Sigma(B) \cap \mathcal{D}$.

Proposition 4.2. *Let $B_0 \in L_{\mathbb{C}}^1(a_1, a_2)$ and $R \geq \|B_0\|_1$. Let $\mathcal{D}$ be an open bounded subset of $\mathbb{C}$ such that its boundary $\mathrm{bd}\,\mathcal{D}$ does not contain quasi-eigenvalues of B_0. Then there exists a neighborhood $W \subset \mathbb{B}_R$ of B_0 in the topology of the metric space $(\mathbb{B}_R, \rho)$ such that, for any $B \in W$,*

there are no quasi-eigenvalues of B on $\mathrm{bd}\,\mathcal{D}$,

the total multiplicity of quasi-eigenvalues of B in $\mathcal{D}$ coincides with that of B_0.

Proof. By Lemmas 3.2 and 4.1, the functional $F(z; B)$ is analytic in z and B and is continuous in the second variable on the metric space $(\mathbb{B}_R, \rho)$. This allows us to apply [8, Theorem 9.17.4] (which is a corollary from Rouché's theorem) to $F(z; B)$. The equivalence $z \in \Sigma(B) \Leftrightarrow F(z; B) = 0$ completes the proof. $\square$

4.2 Resonances' perturbations and directional derivatives of F

Lemma 4.3. *At quasi-eigenvalues $\omega \in \Sigma(B)$, the derivative of F w.r.t. z is given by*

$$\frac{\partial F(\omega; B)}{\partial z} = \frac{2\omega}{\partial_x \theta(a_2, \omega, B)} \int_{a_1}^{a_2} \theta^2(s, \omega; B) B(s) \mathrm{d}s + \frac{i\nu_1}{\partial_x \theta(a_2, \omega, B)} + \frac{\theta(a_2, \omega, B)}{\omega}; \qquad (4.2)$$

and the directional derivatives $[\partial_B F(\omega, B)](V)$ w.r.t. B in the direction $V \in L_{\mathbb{C}}^\infty(a_1, a_2)$ by

$$\frac{\partial F(\omega, B)}{\partial B}(V) = \frac{\omega^2}{\partial_x \theta(a_2, \omega, B)} \int_{a_1}^{a_2} \theta^2(s, \omega; B)\, V(s) \, \mathrm{d}s. \qquad (4.3)$$

The calculation of derivatives resembles that of [21] and is given in Appendix A.2.

Proposition 4.4. *Let ω be a quasi-eigenvalue of $B \in L_{\mathbb{C}}^1$ of multiplicity $m \in \mathbb{N}$. Then:*

(i) For every direction $V \in L_{\mathbb{C}}^1$, there exist $\varepsilon > 0$, $\delta > 0$, and functions Ω_j, $j = 1, \ldots, m$, continuous on $[0, \delta)$ such that for $\zeta \in [0, \delta)$, all the quasi-eigenvalues of $B + \zeta V$ lying in $\mathbb{D}_\varepsilon(\omega)$ are given by $\{\Omega_j(\zeta)\}_1^m$ taking multiplicities into account.

(ii) For each of these functions the following asymptotic formula is valid

$$\Omega_j(\zeta) = \omega + [K(\omega, B; V)\zeta]^{1/m} + o(\zeta^{1/m}) \qquad \text{as } \zeta \to 0, \quad \text{with } K(\omega, B; V) := -\frac{m! \frac{\partial F(\omega, B)}{\partial B}(V)}{\partial_z^m F(\omega; B)}. \qquad (4.4)$$

(iii) In the case $K(\omega, B; V) \neq 0$, each branch of $[\cdot]^{1/m}$ corresponds to exactly one of functions Ω_j. So for small enough $\zeta > 0$, all m values of functions $\Omega_j(\zeta)$ are distinct and simple quasi-eigenvalues of $B + \zeta V$.

Proof. For a one-side open resonator, an analogue of the particular case considered in (iii) was obtained in [21, Proposition 3.5]. Statement (i) is well known for resonances of the Schrödinger equation and can be obtained using the Weierstrass preparation theorem and the Puiseux series theory. We start from these arguments to prove that (4.4) holds always (not only when $K(\omega, B, V) \neq 0$ as in [21]).

Consider the entire function $Q(z, \zeta) := F(z; B + \zeta V)$ of two complex variables z and ζ. Then ω is an m-fold zero of the function $Q(\cdot, 0)$. By the Weierstrass preparation theorem, in a certain polydisc $\mathcal{D}_\varepsilon(\omega) \times \mathcal{D}_{\delta_1}(0)$,

$$Q(z, \zeta) = [(z - \omega)^m + h_1(\zeta)(z - \omega)^{m-1} + \cdots + h_m(\zeta)]R(z, \zeta),$$

where the coefficients h_j (the function $R(z, \zeta)$) are analytic in $\mathcal{D}_{\delta_1}(\omega)$ (resp., in $\mathcal{D}_\varepsilon(\omega) \times \mathcal{D}_{\delta_1}(0)$),

$$h_j(0) = 0, \quad \text{and} \quad R(z, \zeta) \neq 0 \ \text{in} \ \mathcal{D}_\varepsilon(\omega) \times \mathcal{D}_{\delta_1}(0).$$

Differentiating Q by ζ, one gets $\frac{\partial F(\omega, B)}{\partial B}(V) = \partial_\zeta Q(\omega, 0) = h'_m(0)R(\omega, 0)$. On the other side,

$$\partial_z^m F(\omega, B) = \partial_z^m Q(\omega, 0) = m!R(\omega, 0) \neq 0.$$

Hence,

$$h'_m(0) = \frac{\partial_\zeta Q(\omega, 0)}{R(\omega, 0)} = \frac{m! \frac{\partial F(\omega; B)}{\partial B}(V)}{\partial_z^m F(\omega, B)} = -K(\omega, B; V). \tag{4.5}$$

Denote by $P(z, \zeta) := (z - \omega)^m + h_1(\zeta)(z - \omega)^{m-1} + \cdots + h_m(\zeta)$ the Weierstrass polynomial. Since $R(z, \zeta) \neq 0$, the zeros of Q and P in $\mathcal{D}_{\varepsilon_1}(\omega) \times \mathcal{D}_{\delta_1}(0)$ coincide. It is well known that for small enough $\delta \in (0, \delta_1]$ and $|\zeta| < \delta$, there are exactly m z-roots (counting multiplicity) of $P(z, \zeta) = 0$ and these roots are given by a one or more convergent (and possibly multivalued) Puiseux series. We denote these m branches of (one or several) Puiseux series by $\Omega_j(\zeta)$.

The method of the Newton diagram (see e.g. [44] and [38, Theorem XII.2]) implies that, if the roots $\Omega_j(\zeta)$ are given by more than one Puiseux series, then $h'_m(0) = K(\omega, B, V) = 0$ and all the roots have the asymptotics $\Omega_j(\zeta) = \omega + o(\zeta^{1/m})$.

In the case when the roots $\Omega_j(\zeta)$ are given by one Puiseux series, this series can be written in powers of $\zeta^{1/m}$ and formula (4.4) follows from equalities (4.5) and $(\Omega_j - \omega)^m + h_1(\zeta)(\Omega_j - \omega)^{m-1} + \cdots + h_m(\zeta) = 0$, see details in the proof of [21, Proposition 3.5]. $\square$

5 Variational characterization of the nonlinear spectrum

Recall that $\chi_{\mathbb{C}_+}$ is the indicator function of $\mathbb{C}_+$. A function $y \in W_{\mathbb{C}}^{2,1}[a_1, a_2]$ is called a nontrivial solution to

$$y''(x) = -\omega^2 y(x) \left[b_1(x) + [b_2(x) - b_1(x)]\chi_{\mathbb{C}_+}(y^2(x)) \right] \tag{5.1}$$

if $y \not\equiv 0$ and (5.1) is satisfied for a.a. $x \in [a_1, a_2]$. If problem (5.1), (1.2), (1.3) admits a nontrivial solution y, we say that y is an eigenfunction and ω is an eigenvalue of (5.1), (1.2), (1.3). In short, we will say that *ω is a nonlinear eigenvalue. The set of nonlinear eigenvalues is denoted by* Σ^{nl}.

Note that

$$\Sigma^{\mathrm{nl}} \text{ is symmetric w.r.t. } i\mathbb{R}. \tag{5.2}$$

Indeed, if $\omega = i\zeta \in \Sigma^{\mathrm{nl}}$ and y is an associated eigenfunction, then for $B = [b_1 + (b_2 - b_1)\chi_{\mathbb{C}_+}(y^2)]$ one has $\omega \in \Sigma(B)$. Since $\omega_1 = i\bar{\zeta} \in \Sigma(B)$ and $\bar{y}$ is an eigenfunction associated with ω_1 (see Lemma 3.3 (ii) an its proof), we conclude that ω_1 is a nonlinear eigenvalue with $y_1 := i\bar{y}$ as an eigenfunction.

Let us notice also that $B = [b_1 + (b_2 - b_1)\chi_{\mathbb{C}_+}(y^2)]$ belongs to $\mathbb{A}$ and, moreover, belongs to the set ext $\mathbb{A}$ of extreme points of $\mathbb{A}$. So,

$$\Sigma^{\mathrm{nl}} \subset \Sigma[\mathrm{ext}\,\mathbb{A}] \subset \Sigma[\mathbb{A}]. \tag{5.3}$$

Let K be the functional defined in Proposition 4.4 (ii) and let cone $K(\omega, B; \mathbb{A} - B)$ be the nonnegative convex cone generated by the set $K(\omega, B; \mathbb{A} - B) = \{K(\omega, B; \widetilde{B} - B) : \widetilde{B} \in \mathbb{A}\}$.

Theorem 5.1. *Let $\omega \notin i\mathbb{R}$. Then the following statements are equivalent:*

(i) ω is nonlinear eigenvalue,

(ii) there exists $B_0 \in \mathbb{A}$ such that $\omega \in \Sigma(B_0)$ and cone $K(\omega, B_0; \mathbb{A} - B_0) \neq \mathbb{C}$.
   If (i)-(ii) hold true, then there exists an eigenfunction y of (5.1), (1.2), (1.3) such that

$$B_0(x) = \left[b_1(x) + [b_2(x) - b_1(x)]\chi_{\mathbb{C}_+}(y^2(x)) \right] \quad a.e. \ on \ (a_1, a_2). \tag{5.4}$$

The proof (see Section 5.2) relies on the rotation properties of θ described in Section 5.1.

5.1 The turning interval $[x_*, x^*]$ and rotational properties of modes.

By $\arg z$ we understood the multivalued argument function of a complex variable $z \neq 0$. In this subsection, we write $\theta(x)$ for $\theta(x, \omega; B)$.

Lemma 5.2. *Let $B \in L^1$ and $\omega \in \Sigma(B)$. Assume that $\alpha = \mathrm{Re}\,\omega > 0$ and $B(x) \geq 0$ a.e. on (a_1, a_2). Then:*

(i) There exists a unique subinterval $[x_*, x^*]$ of $[a_1, a_2]$ with the properties that

$$\overline{\theta(x)}\partial_x\theta(x) \in \mathbb{C}_- \ when \ x < x_*, \qquad \overline{\theta(x)}\partial_x\theta(x) \in \mathbb{R} \ when \ x \in [x_*, x^*],$$
$$and \quad \overline{\theta(x)}\partial_x\theta(x) \in \mathbb{C}_+ \ when \ x > x^*. \tag{5.5}$$

(It is supposed that $x_* \leq x^*$ and that, in the case $x_* = x^*$, the turning interval $[x_*, x^*]$ degenerates into a single turning point $x_* = x^*$. The meaning of 'turning' is explained by statement (5.7) below).

(ii) If $x_* < x^*$, then

$$B = 0 \ a.e. \ on \ (x_*, x^*) \ and \ \partial_x\theta(x) \ is \ constant \ on \ [x_*, x^*]. \tag{5.6}$$

In particular, $B > 0$ a.e. yields $x_* = x^*$.

(iii) There exists at most one point $x_0 \in [a_1, a_2]$ such that $\theta(x_0) = 0$. If x_0 does exist, it belongs to $[x_*, x^*]$. (In the sequel, when x_0 does not exist, we assume $\{x_0\} = \varnothing$.)

(iv) The multifunction $\arg\theta(x)$ has a branch $\arg_*\theta(x)$ that is defined on $[a_1, a_2] \setminus \{x_0\}$ and has the following properties:

$$\arg_*\theta(x) \ is \ differentiable \ on \ [a_1, a_2] \setminus \{x_0\},$$
$$\partial_x \arg_*\theta < 0 \ if \ x < x_*, \quad \partial_x \arg_* > 0 \ if \ x > x^*, \quad \partial_x \arg_*\theta = 0 \ if \ x \in [x_*, x^*] \setminus \{x_0\}. \tag{5.7}$$

(v) $\partial_x \arg_*\theta(x)$ is bounded on $[a_1, a_2] \setminus \{x_0\}$. In particular, for every $\xi \in [-\pi, \pi)$,

$$the \ set \ \{x \in [a_1, a_2] \setminus [x_*, x^*] \ : \ \arg\theta(x) = \xi(\mathrm{mod}\,2\pi)\} \ is \ at \ most \ finite.$$

Proof. **(i)-(ii)** Since $\operatorname{Im}\omega < 0$ and $\operatorname{Re}\omega > 0$, we have $\operatorname{Im}\omega^2 < 0$ and

$$\partial_x \operatorname{Im}(\bar{\theta}\partial_x\theta) = \operatorname{Im}(\bar{\theta}\partial_x^2\theta) = \operatorname{Im}(-\bar{\theta}\omega^2 B\theta) = B|\theta|^2 \operatorname{Im}(-\omega^2) \geq 0 \text{ a.e.} \tag{5.8}$$

For $x = a_1$ and $x = a_2$ using (3.1) and (1.3), one gets

$$\operatorname{Im}[\overline{\theta(a_1)}\partial_x\theta(a_1)] = \nu_1 \operatorname{Im}(-\mathrm{i}\omega) = -\nu_1\alpha \leq 0$$

and

$$\operatorname{Im}[\overline{\theta(a_2)}\partial_x\theta(a_2)] = \operatorname{Im}\left[\frac{\partial_x\overline{\theta(a_2)}}{-\mathrm{i}\bar{\omega}\nu_2}\partial_x\theta(a_2)\right] = \frac{|\partial_x\theta(a_2)|^2}{\nu_2}\operatorname{Im}\frac{1}{-\mathrm{i}\bar{\omega}} \geq 0.$$

Hence, $\operatorname{Im}(\bar{\theta}\partial_x\theta)$ is a nondecreasing continuous function having a zero in $[a_1, a_2]$. Denote

$$x_* = \min\{x : \operatorname{Im}[\overline{\theta(x)}\partial_x\theta(x)] = 0\} \text{ and } x^* = \max\{x : \operatorname{Im}[\overline{\theta(x)}\partial_x\theta(x)] = 0\}.$$

Then

$$\operatorname{Im}[\overline{\theta(x)}\partial_x\theta(x)] < 0 \text{ for } x < x_*, \quad \text{and} \quad \operatorname{Im}[\overline{\theta(x)}\partial_x\theta(x)] > 0 \text{ for } x > x^*. \tag{5.9}$$

When $x_* < x^*$, (5.8) implies

$$\operatorname{Im}[\overline{\theta(x)}\partial_x\theta(x)] = 0 \text{ for all } x \in [x_*, x^*]. \tag{5.10}$$

Combining this with $\partial_x^2\theta = -\omega^2 B\theta$, we get (5.6).

(iii) It follows from (i) that every x-zero of θ belongs to $[x_*, x^*]$. If there exist two distinct zeroes, then (5.6) yields $\partial_x\theta \equiv 0$ on $[x_*, x^*]$ and so $\theta \equiv 0$ on $[a_1, a_2]$, a contradiction (to (3.1)).

(iv) Consider an arbitrary continuous on $[a_1, a_2] \setminus \{x_0\}$ branch $\arg_* \theta(x)$ of $\arg\theta(x)$. Combining the equality

$$\frac{\operatorname{Im}(\bar{\theta}\partial_x\theta)}{|\theta|^2} = \operatorname{Im}\frac{\partial_x\theta}{\theta} = \operatorname{Im}\partial_x\ln\theta = \partial_x\arg_*\theta \quad \text{for } x \neq x_0, \tag{5.11}$$

with (5.9) and (5.10) we get (iv).

(v) Due to formula (5.11), it is enough to prove that $\partial_x\arg_*\theta$ is bounded in a punctured neighborhood of x_0 assuming that x_0 exists.

To be specific, consider the case $x > x_0$. Since $\theta(x_0) = 0$, one has $\theta'(x_0) \neq 0$ and $\theta(x) = \theta'(x_0)(x - x_0) - \omega^2 \int_{x_0}^x (x - s)\theta(s)B(s)\mathrm{d}s$. Hence,

$$\partial_x\arg_*\theta = \frac{\operatorname{Im}(\bar{\theta}\partial_x\theta)}{|\theta|^2} = |\omega|^4 \int_{x_0}^x \int_{x_0}^x B(s)B(t)\operatorname{Im}\frac{(x - s)\overline{\theta(s)}\theta(t)}{|\theta(x)|^2}\,\mathrm{d}s\,\mathrm{d}t =$$

$$= |\omega|^4 \int_{x_0}^x \int_{x_0}^x B(s)B(t)\operatorname{Im}\frac{(x - s)\overline{\theta(s)}(t - s)\theta'(p(t,s))}{|\theta(x)|^2}\,\mathrm{d}s\,\mathrm{d}t, \tag{5.12}$$

where $p(t, s)$ lies between t and s. Since $\frac{(x-s)(t-s)}{|\theta(x)|^2}$ is bounded in the square $(s, t) \in (x_0, x)^2$ (recall that $\theta'(x_0) \neq 0$) and $\theta, \theta' \in C[a_1, a_2]$, we see that the last integral in (5.12) is uniformly bounded for $x > x_0$. □

Now consider modes θ corresponding quasi-eigenvalues on the axis $\mathrm{i}\mathbb{R}$. In this case, the analysis is slightly different, but simpler, since

$$\theta(x, z; B) \text{ are real when } z \in \mathrm{i}\mathbb{R} \text{ and } B \in L_\mathbb{R}^1(a_1, a_2). \tag{5.13}$$

Lemma 5.3. *Let $\omega \in \Sigma(B)$, $\omega = \mathrm{i}\beta$, $\beta \in \mathbb{R}$, and $B(x) \geq 0$ a.e. on (a_1, a_2). Then:*

(i) *There exists a unique subinterval $[x_*, x^*]$ of $[a_1, a_2]$ such that*

$$\theta(x)\partial_x\theta(x) < 0 \; if \; x < x_*, \qquad \theta(x)\partial_x\theta(x) = 0 \; if \; x \in [x_*, x^*],$$

$$and \quad \theta(x)\partial_x\theta(x) > 0 \; if \; x > x^*.$$

(ii) *If $x_* < x^*$, then $B(x) = 0$ a.e. on (x_*, x^*) and $\theta(x)$ is a nonzero constant function on $[x_*, x^*]$. In particular, if $B(x) > 0$ a.e. on (a_1, a_2), then $x_* = x^*$.*

(iii) *If $\theta(x_0) = 0$, then $x_0 = x_* = x^*$.*

Proof. Let $x_* = \min\{x \in [a_1, a_2] \; : \; \theta(x)\partial_x\theta(x) = 0\}$ and $x^* = \max\{x \in [a_1, a_2] \; : \; \theta(x)\partial_x\theta(x) = 0\}$. The existence of x_* and x^* follows from (1.2)-(1.3) and $\theta\partial_x\theta \in C[a_1, a_2]$. Since

$$\partial_x(\theta\partial_x\theta) = (\partial_x\theta)^2 + \theta\partial_x^2\theta = (\partial_x\theta)^2 + \beta^2 B\theta^2 \geq 0 \;\; \text{a.e. on } (a_1, a_2),$$

we obtain (i). The equality

$$0 = \theta(x^*)\partial_x\theta(x^*) - \theta(x_*)\partial_x\theta(x_*) = \int_{x_*}^{x^*} \partial_x(\theta\partial_x\theta) \; \mathrm{d}x = \int_{x_*}^{x^*} [(\partial_x\theta)^2 + \beta^2 B\theta^2]\mathrm{d}x$$

implies $\partial_x\theta(x) = 0$ and $B(x) = 0$ a.e. on $[x_*, x^*]$. When $x_* < x^*$, this and $\theta \not\equiv 0$ yield statement (ii). Statement (iii) follows from (ii). $\qquad\square$

5.2 Proof of Theorem 5.1

Let $\alpha = \operatorname{Re}\omega \neq 0$. Due to Lemma 3.3 (ii) and (5.2), without loss of generality one can assume $\alpha > 0$. Suppose $\omega \in \Sigma(B_0)$ for certain $B_0 \in \mathbb{A}$. Put

$$C_0 = \frac{\partial_x\theta(a_2, \omega, B_0)}{\omega^2}.$$

By (4.3),

$$C_0\frac{\partial F(\omega; B_0)}{\partial B}(V) = \int_{a_1}^{a_2} \theta^2(s, \omega; B_0) \; V(s) \; \mathrm{d}s \;.$$

Define the set $S_0 \subset \mathbb{C}$ by

$$S_0 := C_0\frac{\partial F(\omega; B_0)}{\partial B}[\mathbb{A} - B_0] = \left\{\int_{a_1}^{a_2} \theta^2(s, \omega; B_0) \; V(s) \; \mathrm{d}s \;\; : \;\; V \in \mathbb{A} - B_0\right\}. \tag{5.14}$$

The definition of $K(\omega, B_0; V)$ in (4.4) implies the equivalence

$$\operatorname{cone} S_0 \neq \mathbb{C} \quad \Longleftrightarrow \quad \operatorname{cone} K(\omega, B_0; \mathbb{A} - B_0) \neq \mathbb{C}. \tag{5.15}$$

Proof of implication (i) $\Rightarrow$(ii). Let y be an eigenfunction of (5.1), (1.2), (1.3) associated with ω. Let us define B_0 by equality (5.4) for all $x \in [a_1, a_2]$. Then $\omega \in \Sigma(B_0)$. Moreover, there exist $C \in \mathbb{R} \setminus \{0\}$ and $\xi_* \in [-\pi, \pi)$ such that $y(x) = Ce^{-i\xi_*/2}\theta(x, \omega; B_0)$. It follows from (5.4) that

$$B_0(x) = \begin{cases} b_1(x), & \text{when} \quad \theta^2(x) \in e^{i\xi_*}\overline{\mathbb{C}_-} \\ b_2(x), & \text{when} \quad \theta^2(x) \in e^{i\xi_*}\mathbb{C}_+ \end{cases}. \tag{5.16}$$

This implies that for every $V \in \mathbb{A} - B_0$ the following assertion hold:

$$V(x) \geq 0 \qquad \text{for a.a. } x \text{ such that } \theta^2(x) \in e^{i\xi_*}\overline{\mathbb{C}_-},$$
$$V(x) \leq 0 \qquad \text{for a.a. } x \text{ such that } \theta^2(x) \in e^{i\xi_*}\mathbb{C}_+.$$

So $\int_{a_1}^{a_2} \theta^2 V \, \mathrm{d}s \in e^{i\xi_*}\overline{\mathbb{C}_-}$ for each $V \in \mathbb{A} - B_0$. Thus, cone $S_0 \subset e^{i\xi_*}\overline{\mathbb{C}_-}$. Equivalence (5.15) concludes the proof of (ii).

Proof of implication (ii) $\Rightarrow$(i). Put $\arg_* \theta^2(x) := 2 \arg_* \theta(x)$. Define the sets $E_s^{\pm} \subset [a_1, a_2]$ by

$$E_s^+ := \mathrm{supp}[b_2 - B_0]_+, \qquad E_s^- := \mathrm{supp}[B_0 - b_1]_+,$$

(for $[\cdot]_+$ see the notation part of Section 1).

Lemma 5.4. *Let* $\mathrm{Re}\,\omega > 0$ *and* $\theta(x) = \theta(x, \omega; B_0)$. *Then*

$$\theta^2[E_s^+] \cup \left(-\theta^2[E_s^-]\right) \subset \overline{\mathrm{cone}\, S_0},$$

i.e., the image of the set E_s^+ *under the function* θ^2 *and the image of the set* E_s^- *under the function* $(-\theta^2)$ *are subsets of the closed cone* $\overline{\mathrm{cone}\, S_0}$

Proof. Recall that the point x_0 was defined in Lemma 5.2 and that, in the case when x_0 does not exist, the notation $\{x_0\}$ means the empty set.

Step 1. If x_0 exists, $\theta(x_0) = 0 \in \overline{\mathrm{cone}\, S_0}$.

Step 2. Suppose $x_1 \in E_s^+ \setminus \{x_0\}$. Then $V_\varepsilon^+(x) := \chi_{(x_1-\varepsilon, x_1+\varepsilon)}(x) \, [b_2 - B_0]_+(x)$ has a nonempty support for every $\varepsilon > 0$. Let us show that (for a suitable choice of values of arg)

$$\arg\left[C_0 \frac{\partial F(\omega; B_0)}{\partial B}(V_\varepsilon^+)\right] - \arg_* \theta^2(x_1) \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.17}$$

By Lemma 5.2 (iv),

$$\arg_* \theta^2 \text{ is defined and continuous in a vicinity of } x_1. \tag{5.18}$$

For small enough $\varepsilon > 0$, $|\arg_* \theta^2(x) - \arg_* \theta^2(x_1)| < \pi/2$ in the interval $x \in (x_1 - \varepsilon, x_1 + \varepsilon)$ and therefore the equality

$$C_0 \frac{\partial F}{\partial B}(V_\varepsilon^+) = \int_{a_1}^{a_2} \theta^2 \, V_\varepsilon^+ \, \mathrm{d}s = \int_{x_1-\varepsilon}^{x_1+\varepsilon} \theta^2[b_2 - B_0]_+ \mathrm{d}s,$$

implies that $C_0 \frac{\partial F}{\partial B}(V_\varepsilon^+) \in \overline{\mathrm{cone}\, \theta^2[(x_1 - \varepsilon, x_1 + \varepsilon)]} \setminus \{0\}$, where $\theta^2[(x_1 - \varepsilon, x_1 + \varepsilon)]$ is the image of $(x_1 - \varepsilon, x_1 + \varepsilon)$ under θ^2. This and (5.18) yields (5.17).

Step 3. Suppose $x_1 \in E_s^- \setminus \{x_0\}$. Then, similar to Step 2, $V_\varepsilon^-(x) := -\chi_{(x_1-\varepsilon, x_1+\varepsilon)}(x)[B_0 - b_1]_+(x)$ has a nonempty support for small ε and $\arg\left[C_0 \frac{\partial F}{\partial B}(V_\varepsilon^-)\right] - \arg_* \theta^2(x_1) - \pi \to 0$ as $\varepsilon \to 0$. In other words,

$$\arg C_0 \frac{\partial F}{\partial B}(V_\varepsilon) - \arg[-\theta^2(x_1)] \to 0 \quad \text{as } \varepsilon \to 0.$$

Step 4. Since $B + V_\varepsilon^{\pm} \in \mathbb{A}$, we see from Steps 2-3 that $\theta^2(x_1) \in \overline{\mathrm{cone}\, S_0}$ whenever $x_1 \in E_s^+ \setminus \{x_0\}$, and $[-\theta^2(x_1)] \in \overline{\mathrm{cone}\, S_0}$ whenever $x_1 \in E_s^- \setminus \{x_0\}$. This completes the proof. $\square$

Lemma 5.5. *Let* $\mathrm{Re}\,\omega > 0$. *Assume that* cone S_0 *is contained in a certain closed half-plane* $e^{i\xi_*}\overline{\mathbb{C}_-}$, *where* $\xi_* \in [-\pi, \pi)$. *Then:*

(i) equality (5.16) holds for a.a. $x \in [a_1, a_2]$,

(ii) $\omega \in \Sigma^{\mathrm{nl}}$ *and* $y = e^{-i\xi_*/2}\theta$ *is an associated eigenfunction of (5.1), (1.2), (1.3).*

Proof. **(i)** Recall that the set E is defined by (2.4). From the definition of E and $B_0 \in \mathbb{A}$, we see that $E \subset X_0 \cup E_s^+ \cup E_s^-$, where X_0 is of zero measure. Clearly, $\overline{\mathrm{cone}\, S_0} \subset e^{i\xi_*}\overline{\mathbb{C}_-}$. By Lemma 5.4,

$$\theta^2[E_s^+] \subset e^{i\xi_*}\overline{\mathbb{C}_-} \qquad \text{and} \qquad \theta^2[E_s^-] \subset e^{i\xi_*}\overline{\mathbb{C}_+}. \tag{5.19}$$

So $E_s^+ \cap E_s^- \subset \{x : \theta^2(x) \in e^{i\xi_*}\mathbb{R}\}$. By Lemma 5.2 (iv)-(v),

$$\text{the set} \quad X_1 := \{x : \theta(x) \in e^{i\xi_*}\mathbb{R}\} \setminus [x_*, x^*] \quad \text{is at most finite.} \tag{5.20}$$

Thus, $E \subset X_0 \cup X_1 \cup [x_*, x^*] \cup (E_s^+ \setminus E_s^-) \cup (E_s^- \setminus E_s^+)$. On the other side, $B_0(x) = b_1(x) = 0$ a.e. on $[x_*, x^*]$ due to Lemma 5.2 (ii). So $X_2 := ([x_*, x^*] \cap E) \setminus (E_s^+ \setminus E_s^-)$ is of zero measure.

Summarizing, we see that the interval $[a_1, a_2]$ can be written as

$$[a_1, a_2] = ([a_1, a_2] \setminus E) \cup X_0 \cup X_1 \cup X_2 \cup (E_s^+ \setminus E_s^-) \cup (E_s^- \setminus E_s^+).$$

On each of the sets of positive measure in the right side, equality (5.16) is fulfilled for a.a. x.

Additional explanations are needed for fact that $\theta^2(x) \in e^{i\xi_*}\mathbb{C}_+$ for a.a. $x \in E_s^- \setminus E_s^+$. Assume that x_1 belongs to the set X_3 consisting of x such that $x \in E_s^- \setminus E_s^+$ and $\theta^2(x) \notin e^{i\xi_*}\mathbb{C}_+$. By (5.19), $\theta^2(x_1) \in e^{i\xi_*}\mathbb{R}$. By (5.20), $X_3 \subset X_1 \cup [x_*, x^*]$. Assume that $\mathrm{meas}\, X_3 > 0$. Then $\mathrm{meas}(X_3 \cap [x_*, x^*]) > 0$. This yields that $\mathrm{meas}\,[\mathrm{supp}[B - b_1]_+ \cap (x_*, x^*)] > 0$. The latter contradicts Lemma 5.2 (ii). This contradiction shows that $\mathrm{meas}\, X_3 = 0$.

(ii) Equality (5.16) implies (5.4). The latter yields (ii). $\qquad\square$

Equivalence (5.15) and Lemmata 5.4, 5.5 imply statement (i) of Theorem 5.1.

The last assertion of Theorem 5.1 follows from Lemma 5.5 (ii) and its proof.

6 Local extremizers and optimization for a fixed frequency

6.1 Definitions of various optimizers

We say that $\alpha \in \mathbb{R}$ is *an admissible frequency* if $\alpha = \mathrm{Re}\,\omega$ for some admissible quasi-eigenvalue ω. So $\mathrm{Re}\,\Sigma[\mathbb{A}]$ is the set of admissible frequencies. Its properties are considered in Subsection 6.2 and Appendix A.3, where it is proved, in particular, that high enough frequencies are admissible at least for most popular settings of quasi-eigenvalue optimization problem.

Definition 6.1 ([23])**.** Let α be an admissible frequency.

(i) *The minimal decay rate $\beta_{\min}(\alpha)$ for the frequency α* is defined by

$$\beta_{\min}(\alpha) := \inf\{\beta \in \mathbb{R} : \alpha - i\beta \in \Sigma[\mathbb{A}]\}.$$

(ii) If $\omega = \alpha - i\beta_{\min}(\alpha)$ is a quasi-eigenvalue for a certain admissible structure $B \in \mathbb{A}$ (i.e., the minimum is achieved), we say that ω and B are of *minimal decay for the frequency α*.

A pair $(\omega, B) \in \mathbb{C} \times L^1$ is called admissible if $B \in \mathbb{A}$ and $\omega \in \Sigma(B)$.

Definition 6.2. We say that an admissible pair (ω_0, B_0) is *a local maximizer of $\mathrm{Im}\,\omega$ for the frequency $\mathrm{Re}\,\omega_0$* if there exist $\varepsilon > 0$ such that $\mathrm{Im}\,\omega_1 \le \mathrm{Im}\,\omega_0$ for any admissible pair (ω_1, B_1) satisfying the following three conditions

$$\mathrm{Re}\,\omega_1 = \mathrm{Re}\,\omega_0, \quad |\omega_1 - \omega_0| < \varepsilon, \quad \|B_1 - B_0\|_1 < \varepsilon.$$
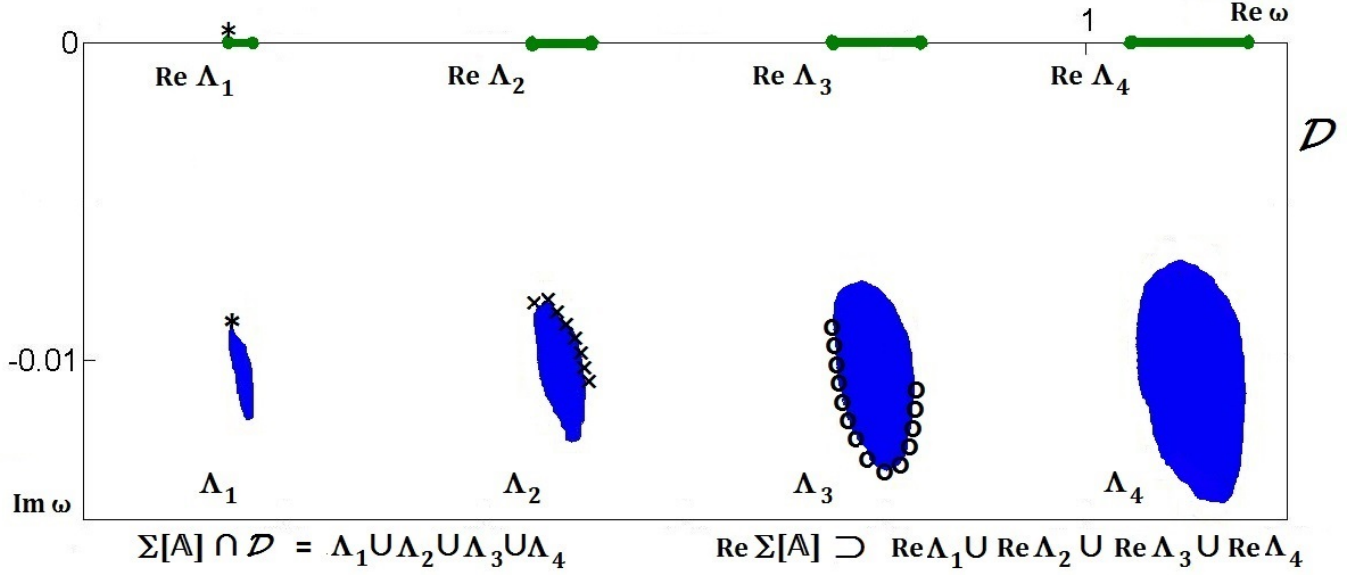
Figure 1: A schematic drawing (see explanations in Remark 6.2) of the part of the set of admissible quasi-eigenvalues $\Sigma[\mathbb{A}]$ in the rectangle $\mathcal{D} = \{\mathrm{Re}\, z \in (0, 1.2),\ \mathrm{Im}\, z \in (-0.015, 0)\}$ and of intervals of admissible frequencies generated by $\Sigma[\mathbb{A}] \cap \mathcal{D}$ for the following choice of parameters: $\nu_1 = 1$, $\nu_2 = +\infty$, $b_1 = 90$, $b_2 = 110$, $a_2 - a_1 = 1$.
'*' mark one of local minimizer for $\beta_{\min}$ and the corresponding quasi-eigenvalue;
'$\times$' mark quasi-eigenvalues of minimal decay for frequencies α in $\mathrm{Re}\,\Lambda_2$;
'o' mark ω-parts of local minimizers of $\mathrm{Im}\,\omega$ for frequencies α in $\mathrm{Re}\,\Lambda_3$.

  Local minimizers of $\mathrm{Im}\,\omega$ for a particular frequency are defined in a similar way. A pair (ω, B) is a *local extremizer* if it is a local minimizer or a local maximizer.

  An admissible frequency α_0 is *a local minimizer for* $\beta_{\min}$ if there exists $\varepsilon > 0$ such that $\beta_{\min}(\alpha_0) \le \beta_{\min}(\alpha)$ for all admissible α in $(\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$. If for certain $\varepsilon > 0$ and all admissible α from a punctured neighborhood $(\alpha_0 - \varepsilon, \alpha_0) \cup (\alpha_0, \alpha_0 + \varepsilon)$ the strict inequality $\beta_{\min}(\alpha_0) < \beta_{\min}(\alpha)$ holds, α_0 is said to be *a strict local minimizer.*

  Usually, in applied and numerical literature, local minimizers of $\mathrm{Im}\,\omega$ are considered without frequency restrictions. It seems that such approach goes back to the definition of [14]. Below we give a rigorous adaptation of that definition suitable for the optimization over $\mathbb{A}$.

  Let us define on the set of admissible pairs (ω, B) the decay rate functional $\mathrm{Dr}(\omega, B) := -\mathrm{Im}\,\omega$.

Definition 6.3. An admissible pair (ω_0, B_0) is said to be a *local minimizer for* Dr if there exist $\varepsilon > 0$ such that $\mathrm{Dr}(\omega_0, B_0) \le \mathrm{Dr}(\omega_1, B_1)$ for every admissible pair (ω_1, B_1) with $|\omega_1 - \omega_0| < \varepsilon$ and $\|B_1 - B_0\|_1 < \varepsilon$. Local maximizers for Dr are defined similarly.

Remark 6.1. Clearly, if (ω_0, B_0) is a local minimizer (maximizer) for Dr, then (ω_0, B_0) is a local maximizer (resp., minimizer) of $\mathrm{Im}\,\omega$ for the frequency $\mathrm{Re}\,\omega_0$.

Remark 6.2. Figure 1 is drawn to illustrate the above definitions of various optimizers. It also provide a rough impression about a part of $\Sigma[\mathbb{A}]$ in $\mathcal{D}$ for the one-side open case $\nu_1 = 1$, $\nu_2 = +\infty$ (cf. Example 6.2). The drawing is partially based on the MATLAB computations of the nonlinear spectrum Σ^{nl} (see Section 11 and Figure 2) and partially on Proposition 6.6, Theorems 8.1, 8.2, 9.1, and formula (10.1).

  In agreement with Theorem 8.2, $\Sigma[\mathbb{A}] \cap \mathcal{D}$ consists of four 'clouds' $\Lambda_1, \ldots, \Lambda_4$. These 'clouds' are connected components of $\Sigma[\mathbb{A}]$ produced by all possible perturbations (inside $\mathbb{A}$) of quasi-eigenvalues $\omega_{0,1} = \omega_0(b_2)$, $\omega_{0,2} = \omega_1(b_2)$, ..., $\omega_{0,4} = \omega_3(b_2)$, resp., which are generated by the homogeneous structure $B_0 \equiv b_2$, see (3.7) and Section 8.

The projections of these components to the real line $\operatorname{Re}\Lambda_j$ are subsets of the set of admissible frequencies $\operatorname{Re}\Sigma[\mathbb{A}]$. However, we *do not know whether* $\bigcup_{j=1}^{4}\operatorname{Re}\Lambda_j$ *are all admissible frequencies in the interval* $(0, 1.2)$. Indeed, it is not known if there exist any admissible quasi-eigenvalues ω with $\operatorname{Re}\omega \in (0, 1.2)$ lying outside of $\mathcal{D}$. One of the difficulties in the study of $\omega \in \Sigma[\mathbb{A}]$ with large $|\omega|$ is that, globally in $\mathbb{C}$, $\Sigma(B)$ is not a continuous set-valued function of B, see Remark 4.1.

In these setting, the frequency 0 is not admissible, see Remark 7.2.

6.2 Admissible frequencies and examples of admissible families

Recall that the notation $\lceil x \rceil$ ($\lfloor x \rfloor$) stands for the ceiling (floor) function.

In this subsection we consider several most reasonable and popular admissible families for which the choices of the constraints has a Physics motivation. For all this families we provide a quantitative version of the statement that *high enough frequencies are admissible.* These estimates on the ranges of admissible frequencies are quite technical. They are derived in Appendix A.3 together with the proof that high enough frequencies are admissible in more general situation, which is described Proposition 6.5 (iii). This proposition also considers other properties of the set $\operatorname{Re}\Sigma[\mathbb{A}]$ of admissible frequencies and, roughly speaking, shows that *the set* $\operatorname{Re}\Sigma[\mathbb{A}]$ *cannot have some 'wild' structure.*

Example 6.1. Let the constraint functions $b_{1,2}$ be constant and $0 < b_1 < b_2$. Assume that $\nu_1 = \nu_2 =: \nu > 0$ and ν^2 equals either b_1, or b_2. These settings represent the admissible family associated with two side open optical cavities of length $a_2 - a_1$ consisting of transparent dielectrics with permittivities in the range from b_1 to b_2 [42]. At the endpoints $a_{1,2}$, the cavity is connected to half-infinite homogeneous outer media consisting of the dielectric material with permittivity ν^2. In Optical Engineering modeling, the outer medium is usually the same as one of dielectrics with extreme allowed permittivities (usually, either $b_1 = \nu^2 = 1$ represents vacuum, or $b_2 = \nu^2$ corresponds to a cavity connected to a waveguide of permittivity b_2 [3]). Under this choice of parameters, formulae (A.9) and (A.13) of Appendix A.3 show that

$$\text{all frequencies in the range } |\alpha| > \frac{\pi}{(a_2 - a_1)b_2^{1/2}} \left\lceil \frac{b_1^{1/2}}{b_2^{1/2} - b_1^{1/2}} \right\rceil \text{ are admissible.}$$

Example 6.2. Let the constraints $b_{1,2}$ be constants and $0 < b_1 < b_2$. Assume that $\nu_1^2 = b_1$ and $\nu_2 = \infty$. One of Optics interpretations of these settings is one-side open cavity, which, in particular, is used for the laser modeling [43]. The Dirichlet boundary condition $y(a_2) = 0$ generated by the parameter $\nu_2 = \infty$ corresponds to a perfect mirror (a plate of a perfect electric conductor perpendicular to the x-axis). So EM waves are reflected without dissipation from the interface plane $x = a_2$. Through the interface plane $x = a_1$ the waves radiate into homogeneous half-infinite medium with permittivity ν_1^2. It was proved in [22] that

$$\text{all frequencies in the range } |\alpha| \geq \frac{\pi}{(a_2 - a_1)b_2^{1/2}} \left(\frac{1}{2} + \left\lceil \frac{b_1^{1/2}}{b_2^{1/2} - b_1^{1/2}} - \frac{1}{2} \right\rceil \right) \text{ are admissible.} \quad (6.1)$$

Example 6.3. In Mechanics models involving the equation for transverse oscillations of a nonhomogeneous string, the linear density $B(x)$ of the string on some intervals may be much less than on others. In such cases, the massless string approximation is employed. This means that $B(x)$ (and so the lower constraint $b_1(x)$) can be equal to 0 on certain intervals. Keeping this in mind, consider constant constraints $b_{1,2} \in [0, +\infty)$ satisfying $0 \leq b_1 < b_2$. Assume that $\nu_1 = 0$ and $0 < \nu_2 < +\infty$. The choice $\nu_1 = 0$ leads to the Neumann boundary condition $y'(a_1) = 0$ and corresponds to the assumption that the left end of the string is "free" (i.e., a massless ring at the end a_1 of the string is sliding without friction on a pole pointing in the transverse direction). The Mechanics interpretation

of the condition (1.3) at a_2 is the linear damping condition (which arises, for instance, if a massless plate immersed in a fluid is connected to the right end a_2). In such settings, ν_2 is the damping coefficient (if the units are chosen such that the tension of the string equals 1). In this context, ν_2 has no natural connection with extreme allowed densities of the string. The estimates of Appendix A.3 shows that for this model

$$\text{all frequencies in the range } |\alpha| \geq \frac{3\pi}{2(a_2 - a_1)(b_2^{1/2} - b_1^{1/2})} \text{ are admissible.}$$

Another interpretation of the Neumann condition $y'(a_1) = 0$, which makes also sense from Optics point of view, is given in the next example.

Example 6.4. Let us assume additionally to the settings of Example 6.1 that $a_2 = -a_1$ and let us restrict optimization to structures symmetric w.r.t. origin, i.e., to B such that $B(x) = B(-x)$ and $b_1 \leq B \leq b_2$ a.e.. This model of a symmetric resonator is used often to reduce computational costs, see e.g. [19]. Since B is an even function and $\nu_1 = \nu_2$, it is easy to see that modes y of such a resonator are either even, or odd, and so either condition $y'(0) = 0$, or the condition $y(0) = 0$ is fulfilled for them. Thus, the quasi-eigenvalue problem and the related optimization problem can be essentially reduced to two: the problem considered in Example 6.3 and a problem similar to that of Example 6.2 (possibly, with different ν_1).

Denote $b_1^{\max} := \operatorname{ess\,sup} b_1(x)$ and $b_2^{\min} := \operatorname{ess\,inf} b_2(x)$. Recall that intervals are, by definition, connected subsets of $\mathbb{R}$ and that an interval called *degenerate* if it consists of one point.

Proposition 6.5. (*i*) *If I is a connected component of $\operatorname{Re}\Sigma[\mathbb{A}]$, then either I is a nondegenerate interval, or $I = \{0\}$. In particular, the only possible isolated admissible frequency is 0.*

(*ii*) *Suppose $b_1^{\max} \leq b_2^{\min}$. If $b_1^{\max} < \nu_1$ or $b_2^{\min} > \nu_2$, then 0 is an admissible frequency.*

(*iii*) *Suppose $b_1^{\max} < b_2^{\min}$. Then high enough frequencies are admissible. Moreover,*

$$\operatorname{Re}\Sigma[\mathbb{A}] \supset \left\{ |\alpha| \geq \frac{2\pi}{(a_2 - a_1)(\sqrt{b_2^{\min}} - \sqrt{b_1^{\max}})} \right\} \tag{6.2}$$

Proof. **(i)** It is enough to show that if $\omega_0 \in \Sigma(B_0)$, $B_0 \in \mathbb{A}$, and $\operatorname{Re}\omega_0 \neq 0$, then B_0 can be piecewise linearly perturbed in $\mathbb{A}$ such that perturbations of ω_0 do not preserve $\operatorname{Re}\omega_0$.

Taking nontrivial $V \in \mathbb{A} - B_0$ consider perturbations $B_0 + \zeta V$ with sufficiently small $\zeta > 0$. If for perturbed quasi-eigenvalues $\omega(\zeta)$ the real part is preserved, we fix a certain branch $\omega(\zeta)$ and certain sufficiently small $\zeta_1 \in (0, 1)$ such that $B_1 = B_0 + \zeta_1 V$ is not an extreme point of $\mathbb{A}$. By Theorem 5.1 and the remarks before it, cone $K(\omega(\zeta_1), B_1; \mathbb{A} - B_1) = \mathbb{C}$. By Proposition 4.4, $\omega(\zeta_1)$ can be perturbed with change of the real part (see also Proposition 10.2).

(ii) follows from Proposition 3.4 (ii).

(iii) For the settings of Example 6.2, (6.2) follows easily from formula (6.1), which was proved in [22] in slightly different form. The proof in general settings can be obtained using the same idea, but requires more technical considerations and involves a number of different cases of mutual arrangements of $\nu_{1,2}$, $b_1^{\max}$, and $b_2^{\min}$. This proof is given in Appendix A.3. $\qquad\square$

6.3 Existence of optimizers and properties of $\Sigma[\mathbb{A}]$

Proposition 6.6. *The set $\Sigma[\mathbb{A}]$ is closed.*

Proof. Since $\mathbb{A} \subset L^1$, the proof requires some modifications of standard compactness arguments. For each $B \in \mathbb{A}$ there exists unique $H(x) = H(x; B)$ such that

$$B(x) = (b_2(x) - b_1(x))H(x) + b_1(x) \text{ a.e. on } (a_1, a_2)$$

and $H(x)$ belongs to the set $\mathbb{A}_*$ of L^∞-functions satisfying

$$0 \le H(x) \le 1 \text{ a.e. on } (a_1, a_2) \text{ and } H(x) = 0 \text{ a.e. on } (a_1, a_2) \setminus E.$$

Let us take $R > \|b_2\|_1$. One can see that

the map $H(\cdot; B) \mapsto B(\cdot)$ is a bijection of $\mathbb{A}_*$ to $\mathbb{A}$ continuous

$$\text{from the weak}^* \text{ topology of } L^\infty \text{ to the metric topology of } (\mathbb{B}_R, \rho) \quad (6.3)$$

(see Subsection 4.1, the continuity is sequential and w.r.t. the corresponding induced topologies on $\mathbb{A}_*$ and $\mathbb{A}$). Lemma 4.1 (i) and the sequential Banach-Alaoglu theorem applied to $\mathbb{A}_*$ complete the proof. $\square$

Proposition 6.6 and $\Sigma[\mathbb{A}] \subset \mathbb{C}_-$ imply three following results.

Corollary 6.7. *(i) For each admissible frequency α, there exist a structure B and the quasi-eigenvalue ω of minimal decay, i.e., $\omega = \alpha - i\beta_{\min}(\alpha)$ belongs to $\Sigma(B)$ for certain $B \in \mathbb{A}$.*

(ii) The pair (ω, B) from statement (i) is a local maximizer of $\operatorname{Im} \omega$ for the frequency α.

Proposition 6.8. *The function $\beta_{\min}$ is lower semicontinuous, i.e., if $\{\alpha_n\}_0^\infty \subset \operatorname{Re} \Sigma[\mathbb{A}]$ and $\alpha_n \to \alpha_0$, then $\beta_{\min}(\alpha_0) \le \liminf \beta_{\min}(\alpha_n)$.*

Proposition 6.9. *If an admissible frequency α is a local minimizer for $\beta_{\min}$, then the pair (ω, B) from Corollary 6.7 is a local minimizer for* Dr.

The existence of local minimizers for $\beta_{\min}$ and Dr is considered in next two sections.

7 Non-uniqueness and local minima for small α

Pure imaginary quasi-eigenvalues $\omega \in i\mathbb{R}$ are the subject of special interest in Mechanics, see e.g. [6, 5]. They corresponds to overdamped and critically damped oscillations.

Proposition 7.1. *If $\operatorname{Re} \omega_0 = 0$ and (ω_0, B_0) is a local extremizer of $\operatorname{Im} \omega$ for the frequency 0, then either $B_0 \equiv b_1$, or $B_0 \equiv b_2$. In particular, if B_0 is a structure of minimal decay for the frequency 0, then either $B_0 \equiv b_1$, or $B_0 \equiv b_2$.*

Proof. Assume that (ω_0, B_0) is a local extremizer of $\operatorname{Im} \omega$ for frequency 0 and that $B_0 \ne b_1$, $B_0 \ne b_2$ (in the sense of L^1-space). Then it follows from (4.3) and Lemma 5.3, that there exist $V_1^\pm \in \mathbb{A} - B_0$ such that $\pm \frac{\partial F(\omega_0; B_0)}{\partial B}(V_1^\pm) > 0$. This, Proposition 4.4, and the symmetry of $\Sigma(B)$ w.r.t. $i\mathbb{R}$ imply that there exist $V_2^\pm \in \mathbb{A} - B_0$ and $\omega_\pm(\zeta) \in \Sigma(B_0 + \zeta V_2^\pm)$ such that for small enough $\zeta > 0$,

$$\operatorname{Re} \omega_\pm(\zeta) = 0, \quad \pm[\operatorname{Im} \omega_\pm(\zeta) - \operatorname{Im} \omega_0] > 0, \quad \text{and} \quad \omega_\pm(\zeta) \to \omega_0 \text{ as } \zeta \to 0$$

(the arguments are similar to [21, Section 4.4], note that V_2^+ and V_2^- are not necessarily different). Thus, (ω_0, B_0) is not a local extremizer. $\square$

Theorem 7.2. *Assume that $\alpha = 0$ is an admissible frequency. Then*

(i) $\Sigma_{\mathrm{im}}(b_1, b_2) := \mathrm{i}\mathbb{R} \cap [\Sigma(b_1) \cup \Sigma(b_2)]$ *is a nonempty closed set.*

(ii) $\beta_{\min}(0) = \min\limits_{\omega \in \Sigma_{\mathrm{im}}(b_1, b_2)} |\mathrm{Im}\,\omega|$

(iii) $\alpha = 0$ *is a strict local minimizer for* $\beta_{\min}$ *whenever* $\omega = -\mathrm{i}\beta_{min}(0)$ *does not belong to* $\Sigma_{\mathrm{mult}}(b_1) \cup \Sigma_{\mathrm{mult}}(b_2)$ *(i.e., whenever the quasi-eigenvalue of minimal decay* ω *is not a multiple quasi-eigenvalue of* b_1 *and of* b_2*).*

Proof. **(i) and (ii)** follow from the assumption $0 \in \mathrm{Re}\,\Sigma[\mathbb{A}]$, Proposition 7.1, and Proposition 6.6.

If 0 is an isolated point of $\mathrm{Re}\,\Sigma[\mathbb{A}]$, statement **(iii)** follows from the definition of a strict local minimizer for $\beta_{\min}$ (see Section 6.1). Let us denote by I the connected component of $\mathrm{Re}\,\Sigma[\mathbb{A}]$ containing 0, and let us prove **(iii)** in the case when $I \neq \{0\}$.

By Lemma 3.3 (ii) and Proposition 6.5 (i), I contains a nonempty symmetric interval $(-\alpha_0, \alpha_0)$. Consider $\beta_* := \liminf_{\alpha \to 0-} \beta_{\min}(\alpha)$. Then also $\beta_* = \liminf_{\alpha \to 0+} \beta_{\min}(\alpha)$ due to Lemma 3.3 (ii).

If $\beta_* = +\infty$, it is obvious that $\alpha = 0$ is a strict minimizer. Consider the case $\beta_* < +\infty$.

Since $\Sigma[\mathbb{A}]$ is closed, there exist sequences $\{B_n\}_1^{+\infty} \subset \mathbb{A}$ and $\omega_{\pm n} \in \Sigma(B_n)$ such that

$$\mathrm{Im}\,\omega_{\pm n} \to -\beta_*, \quad \mathrm{Re}\,\omega_{\pm n} \to 0 \pm 0, \quad \text{and}$$

the sequence B_n converges to a certain $B_* \in \mathbb{A}$ w.r.t. the metric ρ of Subsection 4.1

(this can be proved with the use of the map $H(\cdot; B) \mapsto B(\cdot)$ from (6.3) and the sequential Banach-Alaoglu theorem applied to $\mathbb{A}_*$).

Let us show that $\omega_* := -\mathrm{i}\beta_*$ is a multiple quasi-eigenvalue of B_*. Indeed, $\omega_* \in \Sigma(B_*)$ due to Lemma 4.1 (i). One can take $\varepsilon > 0$ such that $\overline{\mathbb{D}_\varepsilon(\omega_*)}$ does not contain other points of $\Sigma(B_*)$. For large enough n, the disc $\mathbb{D}_\varepsilon(\omega_*)$ contains at least two distinct quasi-eigenvalues $\omega_{\pm n}$ of B_n. Hence, it follows from Proposition 4.2 that the multiplicity of ω_* as a quasi-eigenvalue of B_* is ≥ 2.

Now, the assumption that $\omega = -\mathrm{i}\beta_{\min}(0)$ is a simple quasi-eigenvalue of any corresponding structure B of minimal decay (by Proposition 7.1, B is either b_1, or b_2) yields $\omega \neq \omega_*$ (since they have different multiplicities). So $\beta_* > \beta_{\min}(0)$. This completes the proofs of statement **(iii)**. $\qquad \square$

Now we will show that each of the statements (i)-(iii) of Theorem 7.2 takes place for constant constraints $b_{1,2}$ under the assumption that at least one of the constants $b_1^{1/2}$, $b_2^{1/2}$ is not in the interval $[\nu_1, \nu_2]$.

Recall that the positive function $K_1(b)$ defined in Proposition 3.4 for positive b is given by $K_1(b) = K_1(b, \nu_1, \nu_2) := \frac{1 + \nu_1/\nu_2}{\frac{\nu_1}{b^{1/2}} + \frac{b^{1/2}}{\nu_2}}$. Define

$$K_2(b) = K_2(b, \nu_1, \nu_2) := \mathrm{sgn}[1 - K_1(b)] \left| \frac{1 - K_1(b)}{1 + K_1(b)} \right|^{1/\sqrt{b}} \quad \text{for } b > 0,$$

$$K_2(0) = K_2(0, \nu_1, \nu_2) := \exp(-2/\nu_1 - 2/\nu_2) \quad (\text{when } \nu_1 = 0, \text{ we suppose } K_2(0) := 0).$$

Corollary 7.3. *Let b_1 and b_2 be nonnegative constants such that $b_1^{1/2} < \nu_1$ or $b_2^{1/2} > \nu_2$. Then:*

(i) $\alpha = 0$ *is a strict local minimizer for* $\beta_{\min}$ *and* $\beta_{\min}(0) = -\frac{1}{2(a_2 - a_1)} \ln \max\{K_2(b_1), K_2(b_2)\}$.

(ii) If $K_2(b_1) = K_2(b_2)$, then the set of structures of minimal decay for the frequency 0 consists of two structures, b_1 and b_2.

(iii) If $K_2(b_1) > K_2(b_2)$, then b_1 is the unique structure of minimal decay for the frequency 0.

(iv) If $K_2(b_1) < K_2(b_2)$, then b_2 is the unique structure of minimal decay for the frequency 0.

Remark 7.1. By Proposition 3.4, the quasi-eigenvalue of minimal decay $\omega_0 = -\mathrm{i}\beta_{\min}(0)$ has multiplicity 1 for every associated structure of minimal decay in all the cases of Corollary 7.3 (cf. [5, 20, 21, 23]).

Proof of Corollary 7.3. Proposition 3.4 implies that 0 is an admissible frequency. It follows from (3.8) that $K_2(b) \leq 0$ exactly when $\Sigma(b) \cap i\mathbb{R} = \varnothing$. On the other side, $K_2(b) > 0$ exactly when $\Sigma(b) \cap i\mathbb{R} = \{\omega_0(b)\}$, where $\omega_0(b)$ is defined by (3.7). In this case, $K_2(b) = \exp\left(2(a_2 - a_1)\operatorname{Im}\omega_0(b)\right)$. This, Theorem 7.2, and Proposition 3.4 complete the proof (see also Remark 7.1). $\qquad\square$

Example 7.4 (non-uniqueness of optimizer). Let $C \in \mathbb{R}_+$. Put $b_1(x) \equiv C^2$, $b_2(x) \equiv 4C^2$, $\nu_1 = \nu_2 = C\sqrt{3}$. This correspond to optimization problem for a cavity constrained by $C^2 \leq B(x) \leq 4C^2$ and placed into the outer medium $(-\infty, a_1) \cup (a_2, +\infty)$ with permittivity $3C^2$ (cf. Example 6.1). The length $a_2 - a_1$ is not important for the present example.

Let $\omega_0 := -\frac{i}{2C(a_2 - a_1)} \ln\left(7 + 4\sqrt{3}\right)$. Then the pairs (ω_0, b_1) and (ω_0, b_2) are:

(a) local minimizers for Dr and local maximizers of $\operatorname{Im}\omega$ for the frequency 0.

Moreover,

(b) b_1 and b_2 are structures of minimal decay for the frequency 0,

(c) the frequency $\alpha_0 = 0$ is a strict local minimizer for $\beta_{\min}$.

These statements follow from Corollary 7.3 (i)-(ii) and straightforward calculations. This example answers partially the question of uniqueness of optimal structure discussed in [14, 23, 20]

Remark 7.2 (cf. [7, 21]). Considering the case when constant functions $b_{1,2}$ do not satisfy the assumption of Corollary 7.3, we obtain the following result: if $\nu_1 \leq \sqrt{B(x)} \leq \nu_2$ a.e., then $\Sigma(B) \cap i\mathbb{R} = \emptyset$. Indeed, Theorem 7.2 (i) and Proposition 3.4 imply that $0 \notin \operatorname{Re}\Sigma[\mathbb{A}]$ when $\nu_1 \leq b_1^{1/2} \leq b_2^{1/2} \leq \nu_2$.

8   Local minima in the case of small contrast

Local minimizers for $\beta_{\min}$ and for the decay rate functional Dr were defined in Section 6.1.

Theorem 8.1. *Let $b \in \mathbb{R}_+ \setminus [\nu_1^2, \nu_2^2]$. Then for every natural number N there exists $\varepsilon > 0$ such that for any admissible family $\mathbb{A}$ defined by constant constraints*

$$b_1 \equiv b - \varepsilon_1, \qquad b_2 \equiv b + \varepsilon_2 \quad \text{with } 0 < \varepsilon_1 + \varepsilon_2 < \varepsilon \text{ and } \varepsilon_{1,2} \geq 0,$$

the following statements hold:

(i) In the interval $|\alpha| < \frac{(N+1/2)\pi}{b^{1/2}(a_2 - a_1)}$ there exist at least $2N + 1$ local minimizers for $\beta_{\min}$.

(ii) There exist at least $2N + 1$ local minimizers and at least $2N + 1$ local maximizers for Dr with associated frequencies in the interval $|\alpha| < \frac{(N+1/2)\pi}{b^{1/2}(a_2 - a_1)}$.

Remark 8.1. In the case $b \in (\nu_1^2, \nu_2^2)$, Theorem 8.1 becomes valid if one change the number of extremizers to $2N$ everywhere and the interval to $|\alpha| < \frac{N\pi}{b^{1/2}(a_2 - a_1)}$ (see Figure 1 and Remark 6.2).

The theorem follows from the next (more general) result, its proof, and Proposition 3.4.

Recall that a set $S \subset \mathbb{C}$ is called path-connected if for any two points $z_{1,2} \in S$ there exist a continuous function $f : [0,1] \to S$ such that $f(0) = z_1$ and $f(1) = z_2$.

Theorem 8.2. *Let $B_0 \in L^1$, $B_0(x) \geq 0$ a.e., and let $\mathcal{D}$ be an open bounded subset of $\mathbb{C}$. Assume that $\Sigma(B_0) \cap \mathcal{D}$ consists of $n \in \mathbb{N}$ distinct points $\{\omega_{0,j}\}_{j=1}^n$ and there are no points of $\Sigma(B_0)$ on the boundary $\operatorname{bd}\mathcal{D}$.*

1) Then there exists $\varepsilon > 0$ such that for any functions $b_{1,2}(x)$ satisfying

$$0 \le b_1 \le B_0 \le b_2 \quad a.e. \qquad and \quad 0 < \|b_1 - b_2\|_1 < \varepsilon \tag{8.1}$$

the following statements hold:

(i) $\Sigma[\mathbb{A}] \cap \mathcal{D}$ consists of n (disjoint) closed path-connected components $\Lambda_1, \ldots, \Lambda_n$ satisfying $\omega_{0,j} \in \Lambda_j$, $j = 1, \ldots, n$.

(ii) There exist at least n local minimizers $(\omega_j^{\min}, B_j^{\min})$ and at least n local maximizers $(\omega_j^{\max}, B_j^{\max})$ for Dr such that $\omega_j^{\min(\max)} \in \Lambda_j$, $j = 1, \ldots, n$.

(iii) For each $\alpha \in \cup_1^n \operatorname{Re} \Lambda_j$, there exists at least one local minimizer and at least one local maximizer of $\operatorname{Im} \omega$ for the frequency α.

2) If, additionally, $\mathcal{D}$ has the form $\{\alpha_1 < \operatorname{Re} z < \alpha_2, \quad -\beta_0 < \operatorname{Im} z < 0\}$ with certain $\alpha_{1,2}, \beta_0 \in \mathbb{R}$, and if ε is chosen as in statement (1), then in the frequency interval (α_1, α_2) there exists at least one local minimizer α_0 for $\beta_{\min}$. This minimizer α_0 can be chosen such that $\beta_{\min}(\alpha_0) = \min_{\omega \in \Sigma[A] \cap \mathcal{D}} |\operatorname{Im} \omega|$.

The rest of this section is devoted to the proof of Theorem 8.2.

Proposition 8.3. *Assume that $B_1, B_2 \in L_{\mathbb{C}}^1$ and a bounded open set $\mathcal{D} \subset \mathbb{C}$ are such that for all $t \in [0, 1]$,*

$$\Sigma(B(t)) \cap \operatorname{bd} \mathcal{D} = \varnothing, \qquad where \ B(t) := (1 - t)B_1 + tB_2. \tag{8.2}$$

Then there exist a nonnegative integer n and a family $\{\omega_j(\cdot)\}_1^n$ of continuous complex-valued functions on $[0, 1]$ such that for all $t \in [0, 1]$,

$$\Sigma(B(t)) \cap \mathcal{D} = \{\omega_j(t)\}_1^n \qquad taking \ multiplicities \ into \ account.$$

Proof. Since $F(\cdot; B) \not\equiv 0$ for every $B \in L_{\mathbb{C}}^1$ (see Section 3.1), the total multiplicity of quasi-eigenvalues of B in $\mathcal{D}$ is finite. It follows from assumption (8.2), Proposition 4.2, the Weierstrass preparation theorem, and the Puiseux series theory (see e.g. [38, Theorem XII.2] and [44, 21, 23]) that for every $s \in [0, 1]$ there exist $\varepsilon(s) > 0$, a nonnegative integer n_s, and continuous in $I_s := (s - \varepsilon(s), s + \varepsilon(s))$ functions $\Omega_{s,j}(t)$, $j = 1, \ldots, n_s$, such that $\Sigma(B(t)) \cap \mathcal{D} = \{\Omega_{s,j}(t)\}_{j=1}^{n_s}$ for all $t \in I_s$ (taking multiplicities into account).

It is easy to see that n_s does not depend on s. Let us denote this number n. Choosing a minimal finite subcover from the open cover $\bigcup_{t \in [0,1]} I_t$ of $[0, 1]$, it is easy also to construct (using I_s and $\Omega_{s,j}(t)$) a finite partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval $[0, 1]$ and families $\{\omega_{i,j}(\cdot)\}_{j=1}^n$, $i = 0, \ldots, k - 1$, of continuous on $[t_i, t_{i+1}]$ functions with the property that $\Sigma(B(t)) \cap \mathcal{D} = \{\omega_{i,j}(t)\}_{j=1}^n$ for every $t \in [t_i, t_{i+1}]$ (taking multiplicities into account). Now one can continuously 'glue' these families at $t_1, \ldots, t_{k-1}$ to produce the desired family $\{\omega_j(\cdot)\}_1^n$. $\square$

Proof of Theorem 8.2. Let us take $\delta > 0$ such that the closed discs $\overline{\mathbb{D}_\delta(\omega_{0,j})}$ are pairwise disjoint and all lie in $\mathcal{D}$. By Proposition 4.2, choosing $R > \|B_0\|_1$, we can ensure that there exists a neighborhood W of B_0 in the topology of the metric space $(\mathbb{B}_R, \rho)$ such that for all $B \in W$:

(a) the total multiplicity of quasi-eigenvalues of B in each of the open discs $\mathbb{D}_\delta(\omega_{0,j})$ coincides with the multiplicity of $\omega_{0,j}$ for B_0,

(b) the total multiplicity of quasi-eigenvalues of B in $\mathcal{D}$ coincides with that of B_0,

(c) there no quasi-eigenvalues of B on each of the boundaries $\operatorname{bd} \mathcal{D}$, $\operatorname{bd} \mathbb{D}_\delta(\omega_{0,j})$, $j = 1, \ldots, n$.

Let us take $\varepsilon > 0$ such that $\mathbb{B}_\varepsilon(B_0) \subset W$ and take $b_{1,2}$ satisfying (8.1). Then $\mathbb{A} \subset W$. Denote $\Lambda_j := \Sigma[\mathbb{A}] \cap \overline{\mathbb{D}_\delta(\omega_{0,j})}$. Then Λ_j are closed. By properties (a) and (b), $\Sigma[\mathbb{A}] \cap \mathcal{D} = \cup \Lambda_j$. By (c), $\Lambda_j \subset \mathbb{D}_\delta(\omega_{0,j})$. Proposition 8.3 and the fact that $\mathbb{A}$ is convex imply that each Λ_j is path-connected. This proves **(1.i)**.

Taking $\omega_j \in \Lambda_j$ such that $\operatorname{Im}\omega_j = \max_{\omega \in \Lambda_j} \operatorname{Im}\omega$ and $B_j \in \mathbb{A}$ such that $\omega_j \in \Sigma(B_j)$, one can see that (ω_j, B_j) is a local minimizer for Dr. Since $\mathcal{D}$ is bounded, so are Λ_j. Hence, the existence of local maximizers for Dr can be shown in the same way. This proves **(1.ii)**. The arguments for **(1.iii)** are similar. After all these considerations, statement **(2)** is obvious. $\square$

9 Locally extremal ω are nonlinear eigenvalues

Recall that the set Σ^{nl} of nonlinear eigenvalues was introduced in Section 5. This section is devoted to the following theorem, which states that quasi-eigenvalues optimal in several senses lie in Σ^{nl} .

Theorem 9.1. *A number ω is a nonlinear eigenvalue if at least one of the following conditions hold:*

(i) ω is a quasi-eigenvalue of minimal decay for the frequency $\operatorname{Re}\omega$,

(ii) $\operatorname{Re}\omega \neq 0$ and ω is a boundary point of $\Sigma[\mathbb{A}]$,

(iii) there exists B_0 such that (ω, B_0) is a local extremizer for Dr,

(iv) there exists B_0 such that (ω, B_0) is a local extremizer of $\operatorname{Im}\omega$ for the frequency $\operatorname{Re}\omega$.

The proof given below combines the results of Sections 5-7 with the technique of local boundary points of $\Sigma[\mathbb{A}]$ introduced in [23].

9.1 Local boundary of $\Sigma[\mathbb{A}]$ and the proof of Theorem 9.1

Definition 9.1 (cf. [23])**.** Let $B \in \mathbb{A}$ and $\omega \in \Sigma(B)$. Then:

(i) the complex number ω is called a *B-local boundary point of the image* $\Sigma[\mathbb{A}]$ of $\mathbb{A}$ if ω is a boundary point of $\Sigma[\mathbb{A} \cap W]$ for a certain open neighborhood W of B in the norm topology of L^1.

(ii) ω is called a *B-local interior point* $\Sigma[\mathbb{A}]$ if ω is an interior point of the set $\Sigma[\mathbb{A} \cap W]$ for any open neighborhood W of B (in the norm topology of L^1).

This definition is a particular case of [23, Definition 4.1]. (In [23], such ω are called strongly local boundary points.) Note that, if $B \in \mathbb{A}$, then a complex number $\omega \in \Sigma(B)$ is a *B*-local interior point of $\Sigma[\mathbb{A}]$ exactly when it is not a *B*-local boundary point of $\Sigma[\mathbb{A}]$.

The following statement is a particular case of [23, Theorem 4.1 and Proposition 4.2].

Proposition 9.2 ([23])**.** *Let $B_0 \in \mathbb{A}$ and $\omega \in \Sigma(B_0)$. If $\operatorname{cone} \frac{\partial F(\omega; B_0)}{\partial B}[\mathbb{A} - B_0] = \mathbb{C}$, then ω is a B_0-local interior point of $\Sigma[\mathbb{A}]$.*

Consider the case $\omega \notin i\mathbb{R}$. Combining Proposition 9.2 and Theorem 5.1 with the equivalence (5.15), one easily gets the following result.

Theorem 9.3. *If $\operatorname{Re}\omega \neq 0$ and ω is a B_0-local boundary point of $\Sigma[\mathbb{A}]$ for a certain $B_0 \in \mathbb{A}$, then $\omega \in \Sigma^{\mathrm{nl}}$.*

Now we are able to obtain Theorem 9.1 in the case $\operatorname{Re}\omega = 0$. Indeed, in this case, each of conditions (i)-(iv) implies that ω is a B_0-local boundary point of $\Sigma[\mathbb{A}]$ for a certain B_0. Theorem 9.3 complete the proof.

The proof of Theorem 9.1 for $\omega \in i\mathbb{R}$ follows from Proposition 7.1. Indeed, it is enough to prove that (iv) implies $\omega \in \Sigma^{\mathrm{nl}}$ (see Remark 6.1). Assume that (ω, B_0) is a local extremizer of $\operatorname{Im}\omega$ for the frequency 0. By Proposition 7.1, B_0 coincides either with b_1, or with b_2.

Suppose $B_0 = b_2$. Then $y(x) = e^{i\pi/4}\theta(x, \omega; b_2)$ is a mode associated with the quasi-eigenvalue ω of b_2. By (5.13) and Lemma 5.3 (iii), $\theta^2 > 0$ everywhere on $[a_1, a_2]$ except possibly one point. Hence $y^2(x) \in \mathbb{C}_+$ a.e., and so $b_2 = \left[b_1 + (b_2 - b_1)\chi_{\mathbb{C}_+}(y^2)\right]$ a.e. This means that y is an eigenfunction of (5.1), (1.2), (1.3) associated with the nonlinear eigenvalue ω.

If $B_0 = b_1$, we can put $y(x) = e^{-i\pi/4}\theta(x, \omega; b_1)$. This completes the proof.

10   5-D reduction with the use of bang-bang equations

10.1   Finding of optimizers from the nonlinear spectrum

Assume that we have found the set Σ^{nl} of nonlinear eigenvalues. Then the quasi-eigenvalues of optimal decay can be calculated with the use of the following formula:

$$\beta_{\min}(\alpha) = \min\{-\operatorname{Im}\omega \ : \ \omega \in \Sigma^{\mathrm{nl}} \ \text{ and } \ \operatorname{Re}\omega = \alpha\}. \tag{10.1}$$

(the minimum exists for each $\alpha \in \operatorname{Re}\Sigma[\mathbb{A}]$). This statement follows from (5.3) and Theorem 9.1. As a by-product we get

$$\operatorname{Re}\Sigma[\mathbb{A}] = \operatorname{Re}\Sigma^{\mathrm{nl}}. \tag{10.2}$$

Roughly speaking, *the nonlinear eigenproblem (5.1), (1.2), (1.3) excludes unknown optimal B_0* (in any sense described above) from the optimization problem if corresponding optimal ω_0 is known. In this case, B_0 can be recovered by one of eigenfunctions of the nonlinear problem. Rigorously this is formulated in the following result.

Let us define a map $y \mapsto \mathcal{B}(y)$ from $C[a_1, a_2]$ to $L^1_{\mathbb{R}}(a_1, a_2)$ by

$$\mathcal{B}(y)(\cdot) = b_1(\cdot) + (b_2(\cdot) - b_1(\cdot))\chi_{\mathbb{C}_+}(y^2(\cdot)).$$

Corollary 10.1. *(a) Assume that statement (i) or statement (ii) of Theorem 9.1 hold. Then ω is a quasi-eigenvalue of $B_0 \in \mathbb{A}$ if and only if $B_0 = \mathcal{B}(y)$ a.e. on (a_1, a_2) for a certain eigenfunction y of (5.1), (1.2), (1.3).*

(b) If at least one of the statements (iii), (iv) of Theorem 9.1 hold, then $B_0 = \mathcal{B}(y)$ a.e. on (a_1, a_2) for a certain eigenfunction y of (5.1), (1.2), (1.3).

(c) If $\operatorname{Re}\omega \neq 0$ and ω is a B_0-local boundary point of $\Sigma[\mathbb{A}]$ for a certain $B_0 \in \mathbb{A}$, then $B_0 = \mathcal{B}(y)$ a.e. on (a_1, a_2) for a certain eigenfunction y of (5.1), (1.2), (1.3).

Proof. **(c)** By Proposition 9.2, $\operatorname{cone}\frac{\partial F(\omega_0; B_0)}{\partial B}[\mathbb{A} - B_0] \neq \mathbb{C}$. Combining this and Theorem 5.1, one obtains the desired statement.

In the case $\operatorname{Re}\omega \neq 0$, **(b)** follows from **(c)**. In the case $\operatorname{Re}\omega = 0$, **(b)** easily follows from the arguments at the end of Section 9.1. Note that in the latter case, B_0 coincides either with b_1, or with b_2.

(a) The implication 'if' is obvious. Let us prove the part 'only if' in the case $\operatorname{Re}\omega \neq 0$. Assume that ω is a quasi-eigenvalue of $B_0 \in \mathbb{A}$. Since $\omega \in \operatorname{bd}\Sigma[\mathbb{A}]$, it is a B_0-local boundary point of $\Sigma[\mathbb{A}]$. Thus, (c) implies the desired statement. In the case when $\operatorname{Re}\omega = 0$ and ω is of minimal decay, the implication 'only if' follows from (b). $\qquad\square$

10.2 Nonlinear eigenvalues that are not of minimal decay

Remark 10.1. If B_0 is optimal in one of the senses of Corollary 10.1, then B_0 is an extreme point of $\mathbb{A}$. More precisely, Corollary 10.1 implies that after a possible correction of B_0 on a set of measure zero, $B_0(x)$ satisfies at least one of the equalities $B_0(x) = b_1(x)$, $B_0(x) = b_2(x)$ at every $x \in (a_1, a_2)$. If, additionally, b_1 and b_2 are piecewise constant with a finite number of intervals of constancy (see [21] for the precise formulation in 1-side open case), then B_0 is so. The last statement follows from Lemma 5.2 (v). (For optical problems this means that B_0 represents a multilayer resonator with a finite number of layers.)

Proposition 10.2. *Let* $\operatorname{Re}\omega \neq 0$ *and* $\omega \in \Sigma(B_0)$ *for a certain* $B_0 \in \mathbb{A}$ *that is not an extreme point of* $\mathbb{A}$. *Then* ω *is an interior point of* $\Sigma[\mathbb{A}]$.

The proposition follows from the combination of Proposition 9.2 with Theorem 5.1 and its proof.

Remark 10.2. Applying Proposition 10.2 in the settings of Theorems 8.1 and 8.2, one can see that if $\omega_{0,j} \notin i\mathbb{R}$, than the corresponding path connected component Λ_j (of the set $\Sigma[\mathbb{A}]$) has a nonempty interior.

Example 10.3. Let $B_0 \equiv b \in \mathbb{R}$, $b \neq \nu_{1,2}^2$, and let $\omega_{0,j}$ be equal to $\omega_j(b)$ of (3.7) for $1 \leq j \leq n$. Let ε and $b_{1,2}$ satisfy statement (1) of Theorem 8.2. Then *there exist* $\omega \in \Sigma^{\mathrm{nl}} \setminus i\mathbb{R}$ *that is not a quasi-eigenvalue of minimal decay.*

Indeed, by Remark 10.2, the interiors $\operatorname{int}\Lambda_j$ of Λ_j are not empty. So for any α from their real projection $\operatorname{Re}[\operatorname{int}\Lambda_j]$, Theorem 8.2 and its proof imply the existence of a local minimizer (ω, B) of $\operatorname{Im}\omega$ such that ω is not of minimal decay for α (see Fig. 1 in Section 6)

10.3 Nonlinear eigenvalues as zeros of a function of three variables

Theorem 10.4. *Assume that* $b_{1,2}$ *satisfy (2.3) and*

$$b_1(x) > 0 \quad \text{a.e. on the set } E \text{ (recall that } E = \{x \ : \ b_1(x) \neq b_2(x)\}). \tag{10.3}$$

Let $a \in [a_1, a_2]$ *and* $c_0, c_1 \in \mathbb{C}$. *Then there exists a unique solution* y *to the problem*

$$y(a) = c_0, \qquad y'(a) = c_1 \tag{10.4}$$

for the nonlinear equation

$$y''(x) = -z^2 y(x) \left[b_1(x) + [b_2(x) - b_1(x)]\chi_{\mathbb{C}_+}(y^2(x)) \right]. \tag{10.5}$$

The proof includes a technical part that considers a number of cases for placing of $c_{0,1}$ in $\mathbb{C}$. It is postponed to Appendix A.4.

Example 10.5 (Nonuniqueness). If assumption (10.3) is dropped, then the statement of Theorem 10.4 is incorrect. To see this, consider the case when $b_{1,2}$ are constants and $0 = b_1 < b_2$. Let $z^2 \in \mathbb{C}_-$, $a = a_1$, $c_0 > 0$, and $c_1 = 0$. Then, for small enough positive ε, the functions y_ε defined by $y_\varepsilon(x) = c_0$ for $x \in [a_1, a_2 - \varepsilon]$ and by $y_\varepsilon(x) = c_0 \cos(zb_2[x - a_2 + \varepsilon])$ for $x \in [a_2 - \varepsilon, a_2]$ are solutions to (10.4), (10.5) on $[a_1, a_2]$.

We assume (10.3) in the rest of this section. Assuming also $\xi \in (0, \pi]$, let us define the function $\Theta(x; \xi, z)$ as the solution to (10.5) satisfying the initial conditions

$$\Theta(a_1; \xi, z) = e^{i\xi}, \quad \partial_x \Theta(a_1; \xi, z) = -iz\nu_1 e^{i\xi}. \tag{10.6}$$

Note that Θ satisfies (1.2) for any ξ (with $\omega = z$). Plugging Θ into the second boundary condition (1.3), let us introduce the complex-valued function

$$F_{\mathrm{nl}}(\xi, z) := \Theta(a_2; \xi, z) + \frac{\mathrm{i}\partial_x \Theta(a_2; \xi, z)}{z\nu_2} \qquad \text{for } z \in \mathbb{C} \setminus 0,$$

and $F_{\mathrm{nl}}(\xi, 0) := 1 + \frac{\nu_1}{\nu_2}$ (the logic of this definition is the same as in Section 3.1).

The following corollary essentially reduce the problem of finding of Σ^{nl} (and so, of finding the function $\beta_{\min}$) to a finite-dimensional question of finding of zeros for a function of three real variables. Consider the fiber of the function F_{nl} over 0,

$$F_{\mathrm{nl}}^{-1}(0) := \{(\xi, z) \in (0, \pi] \times \mathbb{C} \ : \ F_{\mathrm{nl}}(\xi, z) = 0\},$$

and the projection of $F_{\mathrm{nl}}^{-1}(0)$ to the z-plane

$$\mathrm{pr}_z \, F_{\mathrm{nl}}^{-1}(0) := \{z \in \mathbb{C} \ : \ F_{\mathrm{nl}}(\xi, z) = 0 \text{ for certain } \xi \in (0, \pi]\}.$$

Corollary 10.6. *Suppose (10.3). Then* $\Sigma^{\mathrm{nl}} = \mathrm{pr}_z \, F_{\mathrm{nl}}^{-1}(0)$.

Proof. The inclusion $\supset$ is obvious. Let us prove $\subset$. Assume that $\omega \in \Sigma^{\mathrm{nl}}$ and y is an associated eigenfunction of (5.1), (1.2), (1.3). Then (1.2) and Theorem 10.4 imply $y(a_1) \neq 0$. Consider two functions $y_\pm(x) := \pm y(x)/|y(a_1)|$. They are also eigenfunctions of (5.1), (1.2), (1.3). (Note that $\mathcal{B}(y) = \mathcal{B}(y_-) = \mathcal{B}(y_+)$). For one of the functions $y_\pm$ (let us denote it by y_1), the equality $y_1(a_1) = e^{\mathrm{i}\xi}$ holds with certain $\xi \in [0, \pi)$. By Theorem 10.4, this function coincide with $\Theta(\cdot; \xi, \omega)$. So $\Theta(\cdot; \xi, \omega)$ is an eigenfunction of (5.1), (1.2), (1.3). Thus, (1.3) yields $F_{\mathrm{nl}}(\xi, \omega) = 0$. $\qquad\square$

For the particular case considered in Example 6.2 and $\omega \notin \mathrm{i}\mathbb{R}$, Corollary 10.6 was announced without proof in [22].

11 A numerical experiment for constant side constraints

To check applicability of the method of Section 10.3 to calculation of quasi-eigenvalues of minimal decay, we take $\nu_1 = 1$, $\nu_2 = +\infty$, $a_1 = -1$, $a_2 = 0$, and the constant functions $b_1 = 90$, $b_2 = 110$. Then $F_{\mathrm{nl}}(\xi, z) = \Theta(0; \xi, z)$ and $\Sigma^{\mathrm{nl}} = \{z \in \mathbb{C}_- \ : \ F_{\mathrm{nl}}(\xi, z) = 0 \text{ for certain } \xi \in (0, \pi]\}$.

To find Σ^{nl} numerically, we take small ε and approximate the sub-level set $\mathcal{Z}_{\mathrm{nl}}(\varepsilon) := \{z \in \mathbb{C} \ : \ \inf_{\xi \in (0,\pi]} |F_{\mathrm{nl}}(\xi)| \leq \varepsilon\}$ by its discrete versions $\mathcal{Z}(h_R, h_I, h_{\mathrm{arg}}; \varepsilon)$ consisting of numbers $z = z_1 + \mathrm{i}z_2$ with $z_1 \in h_R\mathbb{Z}$, $z_2 \in h_I\mathbb{Z}$, such that $\max_{1 \leq n \leq N} |F_{\mathrm{nl}}(nh_{\mathrm{arg}}, z)| \leq \varepsilon$. The angular step h_{arg} is connected with N by $h_{\mathrm{arg}} = \pi/N$.

The value of $F_{\mathrm{nl}}(\xi, z) = \Theta(0; \xi, z)$ is computed by the shooting method applied to the solution Θ of the equation

$$y''(x) = -z^2 y(x) \left[b_1 + [b_2 - b_1]\chi_{\mathbb{C}_+}(y^2(x)) \right]. \tag{11.1}$$

This does not require finite-difference approximation. Indeed, for $z^2 \in \mathbb{C}_-$, the equation (11.1) has constant coefficients on each interval where Θ^2 stays in one of half-planes $\mathbb{C}_\pm$ and admit an explicit analytic solution depending only on the initial values in the left end of this interval. These values and the lengths of intervals of constancy can be computed iteratively starting from initial conditions (10.6).

The result of such a computation of $\mathcal{Z}_{\mathrm{nl}}(\varepsilon)$ with the level $\varepsilon = 5 \times 10^{-5}$ in the domain $\mathcal{D} = \{\mathrm{Re}\, z \in (0, 1.2), \, \mathrm{Im}\, z \in (-0.015, 0)\}$ are plotted in Figure 2 (a). We take $N = 360$. The other step parameters are changed through $\mathcal{D}$ from $(h_R, h_I) = (2.4 \times 10^{-4}, 5.5 \times 10^{-5})$ to $(h_R, h_I) = (2.7 \times 10^{-3}, 2.3 \times 10^{-4})$ to provide better resolution for the regions where $\mathcal{Z}_{\mathrm{nl}}(\varepsilon)$ is concentrated and to reduce computation

time in the regions where the values of function $\mathcal{M}(z) := \max_{1 \le n \le N} |F_{\mathrm{nl}}(nh_{\mathrm{arg}}, z)|$ are essentially greater than ε.

Comparing the shape of this approximation of $\Sigma^{\mathrm{nl}} \cap \mathcal{D}$ with the fact that $\mathrm{bd}\,\Sigma[\mathbb{A}] \setminus i\mathbb{R} \subset \Sigma^{\mathrm{nl}}$ (see Theorem 9.1 (ii)) and Theorems 8.1-8.2, we conclude that the considered example fits into the description of the small contrast case at least in the domain $\mathcal{D}$, and that $\mathcal{Z}_{\mathrm{nl}}(\varepsilon)$ provides an approximate shape not only for $\Sigma^{\mathrm{nl}} \cap \mathcal{D}$, but also for $\mathrm{bd}\,\Sigma[\mathbb{A}] \cap \mathcal{D}$. The form of $\mathrm{bd}\,\Sigma[\mathbb{A}] \cap \mathcal{D}$ together with Theorem 8.2 makes it possible to find approximately the part $\Sigma[\mathbb{A}] \cap \mathcal{D}$ of the set of admissible resonances, which consists of four quasi-eigenvalue 'clouds' $\Lambda_1, \ldots, \Lambda_4$ (see Remark 6.2).
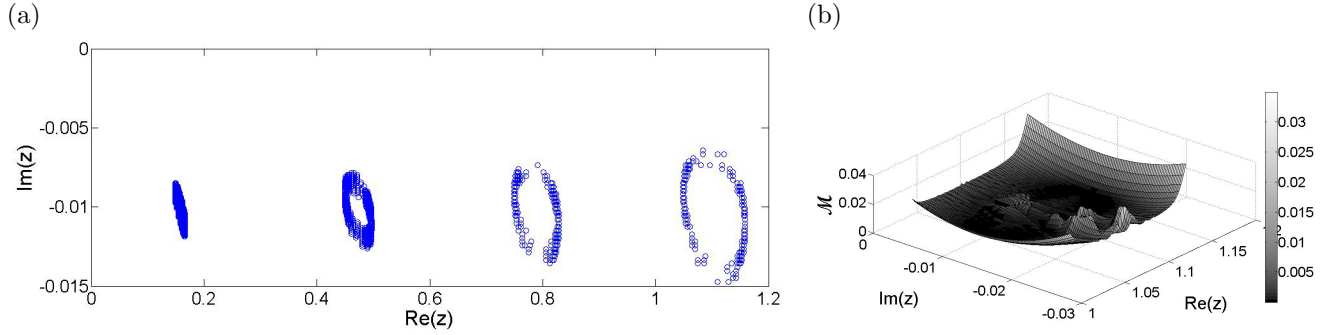
(a) (b)

Figure 2: MATLAB computation of: (a) Σ^{nl} in the rectangle $\mathcal{D}$, (b) the graph of $\mathcal{M}(z) := \max_{1 \le n \le N} |F_{\mathrm{nl}}(nh_{\mathrm{arg}}, z)|$ in the neighborhood $\mathcal{D}_4 = \{\mathrm{Re}\,z \in (1, 1.2),\ \mathrm{Im}\,z \in (-0.025, 0)\}$ of Λ_4.

While the shape of $\mathcal{Z}_{\mathrm{nl}}(\varepsilon)$ in the rectangular neighborhood $\mathcal{D}_4 = \{\mathrm{Re}\,z \in (1, 1.2),\ \mathrm{Im}\,z \in (-0.025, 0)\}$ still resembles a boundary of a closed bounded path-connected component Λ_4 of $\Sigma[\mathbb{A}]$ (and even suggests that Λ_4 is simple connected), the graph of function $\mathcal{M}(z)$ exhibits a close to periodic pattern of ridges and canyon in the regions with $\mathrm{Im}\,z$ close to -0.007 and -0.013, see Figure 2 (b). For the study of Λ_4, the value of $h_{\mathrm{arg}} = \pi/360$ is too large. When h_{arg} is decreased to $\frac{\pi}{360} \times 10^{-3}$ while keeping $h_R \ge 10^{-3}$ and $h_I \ge 10^{-5}$, only one sharp canyon that corresponds to $\mathrm{bd}\,\Sigma[\mathbb{A}] \cap \mathcal{D}_4$ remains.

The values of the minimal decay rate function $\beta_{\min}$ for α in the intervals $\mathrm{Re}\,\Lambda_1, \ldots, \mathrm{Re}\,\Lambda_4$ can be computed using (10.1). We have done these computations for some of the values of α, see Table 1. The step h_I is 5×10^{-7} for the first and second values of α and $h_I = 5 \times 10^{-6}$ for the others, h_{arg} varies from $\frac{\pi}{180} \times 10^{-2}$ to $\frac{\pi}{360} \times 10^{-3}$.

α	0.14977	0.14983	0.16557	0.44931	0.46050	0.49674
$\beta_{\min}(\alpha)$	0.009119	0.009120	0.01115	0.00910	0.00846	0.01115

α	0.74884	0.77200	0.82790	1.04838	1.08800	1.15905
$\beta_{\min}(\alpha)$	0.00910	0.00766	0.01113	0.00909	0.00689	0.01109

Table 1: Some of values of the minimal decay rate function $\beta_{\min}(\alpha)$.

For $\alpha = 1.088$, we have found the structure B generating the quasi-eigenvalue of minimal decay $\alpha - i\beta(\alpha)$. It consists of 7 layers. With $x_0 = -1$ and $x_7 = 0$ for the endpoints of the interval $[a_1, a_2] = [-1, 0]$, the computed intervals of constancy of B can be written in terms of the boundaries of the layers (switch points) x_j, $j = 1, \ldots, 6$. Namely, $B(x) = 110$ for $x \in (x_{2k}, x_{2k+1})$, $k = 0, \ldots, 3$, and $B(x) = 90$ for $x \in (x_{2k+1}, x_{2k+2})$ for $k = 0, \ldots, 2$. Approximate values $\widetilde{x}_j$ for x_j are given in Table 2 together with absolute error estimates ε_j.

j	1	2	3	4	5	6
$\widetilde{x}_j$	-0.86237855	-0.71669445	-0.57371202	-0.43130025	-0.28563822	-0.15697427
ε_j	0.00000028	0.00004800	0.00000900	0.00007000	0.00002000	0.00011200

Table 2: Switch points and their absolute error estimates for $B(x)$ of minimal decay for $\alpha = 1.088$.

One of possible improvements of the proposed method may be concerned with calculation of zeros of the function $F_{\mathrm{nl}}(\xi, z)$. Indeed, we used the simplest straightforward approach based on sub-level sets of the function $\mathcal{M}$. More elaborated approaches that do not neglects the information contained in $\arg F_{\mathrm{nl}}(nh_{\mathrm{arg}}, z)$ should be more efficient.

12 Discussion

Earlier Engineering and Numerical Optimization papers were concerned mainly with the case when the constraint functions b_1 and b_2 are constants. In this section, we (sometimes speculatively) compare our results for this case with earlier suggestions for optimal designs and with discussions of the proper statement of the problem.

12.1 Alternating almost periodic structures with a centered defect.

Contemporary designs of high-Q optical cavities usually involve incorporation of defects into a periodic structure composed of two materials [1, 33, 26]. For cavities with 1-D geometry [19, 33, 26, 3], the base periodic structure consists of alternating layers (say with widths ℓ_1 and ℓ_2) of two materials with different permittivities (say b_1 and b_2, respectively). The Physics justification for such designs is that the periodic structure forms a distributed Bragg reflector with high reflectivity for certain bands of frequencies (stopbands). When a defect is introduced, the waves with such frequencies are expected to accumulate and to be well confined in the region surrounding the defect [17, Chapter 4], [32]. If the fabrication process involves material with permittivities from b_1 to b_2, Remark 10.1 implies that the best confinement has to be produced by structures B that consist only of layers with $B(x) = b_1$ and $B(x) = b_2$ (for one-side open case, see [21, 22]). In other words, both the base structure and the defect have to consist of the two materials with extreme allowed permittivities.

Basing on various experimental and computational approaches, two types of designs with defects were proposed:

(a) a localized defect in a center of a periodic structure, e.g., [1, 19, 9],

(b) the size-modulated 1-D stack [33, 26, 3].

Designs of type (b) assumes gradually changing deviations from periodicity in the widths of layers.

For the structure of minimal decay computed in Section 11 (see Table 2), one can observe that it is *close to alternating periodic with a period consisting of one layer with $B(x) = b_1$ and the other layer with $B(x) = b_2$* (since the structure is finite, either the first, or the last layer has to be treated separately). The widths of layers $[x_1, x_2]$, $[x_3, x_4]$, $[x_5, x_6]$ with $B = b_1$ *gradually decrease from* 0.146 *to* 0.129, and the widths of layers $[x_{2k}, x_{2k+1}]$, $k = 0, \dots, 3$, corresponding to $B = b_2$ *gradually increase from* 0.138 *to* 0.157. This resembles designs of type (b).

Let us note that the 1-D structures considered in [19, 33, 26, 3] are symmetric. If we consider the numerical experiment of Section 11 in the context of optimization of odd modes of a symmetric structure in the interval $(-1, 1)$ (cf. Example 6.4), then the structure B has to be extended to $(0, 1)$ by $\widetilde{B}(\pm x) = B(x)$, $x \in (-1, 0)$. Then the layer $[x_6, -x_6]$ is approximately twice larger of the nearby layers and so the structure does not resemble the size-modulated 1-D stack designs of [33, 26, 3].

The symmetric structure $\widetilde{B}$ can be considered as a *combination of designs (a) and (b)*. It has *a localized defect consisting of the central layer $[x_6, -x_6]$, and this defect is surrounded by alternating layers with gradually changing widths.*

In [33, 26, 3], the quadratic deviation laws were employed to vary the widths of layers. From the point of view of analytic optimization, it would be interesting to find dependence of the base periodic structure, the parameters of the defect, and width deviations on the frequency of corresponding optimal resonance.

12.2 Global minimization of decay rate.

Well-posedness of global maximization for Q-factor related functionals (without frequency restrictions) was discussed in [19, 29]. For 1-D resonators with constant side constraints and fixed length $a_2 - a_1$, the global maximizer for Q-factor does not exist. Indeed, if we take a homogeneous structure $B(x) = b$ (with constant $b \in [b_1, b_2]$, $b \neq \nu_{1,2}^2$), then (3.7) implies that $Q(\omega_n) = -\frac{\mathrm{Re}\,\omega_n}{2\,\mathrm{Im}\,\omega_n}$ go to $+\infty$ as $|n| \to \infty$.

Below, basing our intuition on the numerical experiment of Section 11, we discuss the question of global minimization of the decay rate $|\mathrm{Im}\,\omega|$ (cf. [15]).

It can be seen from Figure 2 (a) that quasi-eigenvalue 'clouds' corresponding to higher frequencies are 'wider'. It is natural to expect that for $|\mathrm{Im}\,\omega|$ great enough they intersect each other and form one large unbounded cloud. This conjecture is supported also by our study of clouds' projections (see Section 6.2 and Appendix A.3) and by the results on existence of multiple quasi-eigenvalues (see Section 3.2). In particular, we already know that the minimal decay function is defined for large enough $|\alpha|$ (Proposition 6.5 (iii)).

It also can be seen from Figure 2 (a) and Table 1 that each of subsequent clouds Λ_1, ..., Λ_4, comes closer to the real line than previous. (The values $\beta(0.14977) \approx 0.009119$, $\beta(0.4605) \approx 0.00846$, $\beta(0.772) \approx 0.00766$, and $\beta(1.088) \approx 0.00689$ approximately corresponds to minima of decay rate for the clouds Λ_1, ..., Λ_4.) Basing on the above observations, we propose for the case when $b_{1,2}$ are constants and $0 \leq b_1 < b_2 < +\infty$ the following

Conjecture 1. The inequality $\liminf_{|\alpha| \to \infty} \beta(\alpha) < \beta(\alpha_0)$ holds for all admissible frequencies α_0. (In particular, the global infimum of the decay rate $\inf_{\omega \in \Sigma[\mathbb{A}]} |\mathrm{Im}\,\omega|$ is not achieved over $\mathbb{A}$.)

A stronger form of this conjecture that assumes existence of $\lim_{|\alpha| \to \infty} \beta(\alpha)$ could help to explain the sliding effects in gradient ascent numerical experiments of [15, 19]. Namely, in these experiments, iteratively improved resonances $\omega^{[n]}$ were approaching to the real axis with simultaneous growth of frequency $\mathrm{Re}\,\omega^{[n]}$. This could happen if $\omega^{[n_0]}$ for some iteration n_0 reach (or come very close to) the Pareto optimal frontier $\{\alpha - \mathrm{i}\beta(\alpha) \ : \ \alpha \in \mathrm{Re}\,\Sigma[\mathbb{A}]\}$ and then move along this frontier to ∞ with growing $\mathrm{Re}\,\omega$ and decreasing $|\mathrm{Im}\,\omega|$. In our opinion, the comparison of results of simulations [3, 15, 19, 33] suggests that $\lim_{|\alpha| \to \infty} \beta(\alpha) = 0$.

It is also interesting to consider the value of $\liminf_{|\alpha| \to \infty} \beta(\alpha)$ from the point of view of the question of uniform separation of the set of resonances $\Sigma(B)$ from the real line (the question of exponential energy decay in the sense of [6, 7, 5]). The absence of quasi-eigenvalues in the certain strip $\{-\beta_0 < \mathrm{Im}\,z < 0\}$ is presently known only under additional conditions involving bounds on total variation of B [7]. So it is difficult to expect that, for certain $\beta_0 > 0$, the set $\Sigma[\mathbb{A}]$ does not intersects with the strip $\{-\beta_0 < \mathrm{Im}\,z < 0\}$ in the case of the admissible family $\mathbb{A}$ restricted only by constant side constraints.

A Appendix

A.1 Analyticity, the proof of Lemma 3.2

(i) follows from (ii) and (3.4). **Let us prove (ii)** for $(z, B) \mapsto \psi(\cdot, z; B)$. Consider the Maclaurin series (in $z \in \mathbb{C}$ and $B \in L^1_{\mathbb{C}}$)

$$\Psi(x, z; B) = (x - a_1) - \psi_1(x; B)z^2 + \psi_2(x; B)z^4 - \psi_3(x; B)z^6 + \dots, \tag{A.1}$$

$$\psi_0(x; B) \equiv x - a_1, \quad \varphi_j(x; B) = \int_{a_1}^x (x - s)\,\varphi_{j-1}(s; B)\,B(s)\mathrm{d}s, \quad j \in \mathbb{N}, \tag{A.2}$$

as a $C[a_1, a_2]$-valued series. One can see that

$$|\psi_j(x; B)| \leq \psi_j(x; |B|) \leq \psi_j(a_2; |B|) \quad \text{for all } x \in [a_1, a_2]. \tag{A.3}$$

It can be shown in the way similar to [18, Sect. 2] that

$$\psi_j(a_2; |B|) \leq \frac{2^j (a_2 - a_1)^{j+1} \|B\|_1^j}{(2j)!}, \quad j \in \mathbb{N}. \tag{A.4}$$

Hence, the series (A.1) converge uniformly on every bounded set of $\mathbb{C} \times L^p_{\mathbb{C}}$. So (A.1) defines an analytic map from $\mathbb{C} \times L^p_{\mathbb{C}}$ to $C[0, \ell]$. Since (A.1) satisfies $\partial_x^2 y = -z^2 By$, $y(a_1) = 0$, and $\partial_x y(a_1) = 1$, we see that $\Psi(x, z; B) = \psi(x, z; B)$. The equality $\partial_x^2 \psi = -z^2 B\psi$ yields that $\partial_x^2 \psi \in L^p_{\mathbb{C}}$ and, moreover, that $(z, B) \mapsto \partial_x^2 \psi(\cdot, z; B)$ is an analytic map from $\mathbb{C} \times L^p_{\mathbb{C}}$ to $L^p_{\mathbb{C}}$. Hence, (A.1) defines an analytic map from $\mathbb{C} \times L^p_{\mathbb{C}}$ to $W^{2,p}$. This map is also bounded-to-bounded due to (A.3), (A.4) and the equalities $\partial_x^2 \psi = -z^2 B\psi$, $\psi(a_1) = 1$.

Similarly one can prove that $(z, B) \mapsto \phi(\cdot, z; B)$ is an analytic and bounded-to-bounded map from $\mathbb{C} \times L^p_{\mathbb{C}}$ to $W^{2,p}$ (cf. [18, 23]). Finally, since $\theta = \varphi - iz\nu_1\psi$, we obtain the same statement for the map $(z, B) \mapsto \theta(\cdot, z; B)$. This completes the proof.

A.2 Directional derivatives, the proof of Lemma 4.3

Differentiation w.r.t. B. For any $z \in \mathbb{C}$, the functions $\theta(x, z; B)$ and $\partial_x \theta(x, z; B)$ satisfy (3.2) and

$$\partial_x \theta(x, z; B) = -iz\nu_1 - z^2 \int_{a_1}^x B(s)\,\theta(s, z; B)\,\mathrm{d}s. \tag{A.5}$$

To find directional derivatives $\frac{\partial \theta}{\partial B}(V)$ and $\frac{\partial\,\partial_x\theta}{\partial B}(V)$, we differentiate these equalities by definition using Lemma 3.2. We get

$$\frac{\partial \theta(x, z; B)}{\partial B}(V) = -z^2 \int_{a_1}^x (x - s)B(s)\frac{\partial \theta(s, z; B)}{\partial B}(V)\mathrm{d}s - z^2 \int_{a_1}^x (x - s)V(s)\theta(s, z; B)\mathrm{d}s, \tag{A.6}$$

$$\frac{\partial\,\partial_x\theta(x, z; B)}{\partial B}(V) = -z^2 \int_{a_1}^x B(s)\frac{\partial \theta(s, z; B)}{\partial B}(V)\mathrm{d}s - z^2 \int_{a_1}^x V(s)\theta(s, z; B)\mathrm{d}s. \tag{A.7}$$

It follows from (A.6) that $\frac{\partial \theta(x, z; B)}{\partial B}(V)$ is the solution $y(x)$ of the initial value problem

$$y''(x) + z^2 B(x)y(x) = f(x), \quad y(a_1) = 0, \; y'(a_1) = 0 \tag{A.8}$$

with $f(x) = -z^2 V(x)\theta(x, z; B)$. Further, (A.7) can be rewritten as $\frac{\partial\,\partial_x\theta(x, z; B)}{\partial B}(V) = y'(x)$ (as a by-product, we get $\frac{\partial\,\partial_x\theta(x, z; B)}{\partial B}(V) = \partial_x\left[\frac{\partial \theta(x, z; B)}{\partial B}(V)\right]$). Solving (A.8) by variation of parameters, one

can find y, y', and, in turn, $y(x) + \frac{iy'(x)}{z\nu_2}$. For $z \neq 0$, $\left[y(x) + \frac{iy'(x)}{z\nu_2} \right]_{x=a_2}$ equals

$$- \left[\theta(a_2) + \frac{i}{z\nu_2} \partial_x \theta(a_2) \right] \int_{a_1}^{a_2} f(s) \ \psi(s, z; B) \ \mathrm{d}s + \left[\psi(a_2) + \frac{i}{z\nu_2} \partial_x \psi(a_2) \right] \int_{a_1}^{a_2} f(s) \ \theta(s, z; B) \ \mathrm{d}s.$$

Substituting f, and $z = \omega \in \Sigma(B)$, we get

$$\theta(a_2, \omega; B) \quad + \quad \frac{i}{\omega\nu_2} \partial_x \theta(a_2, \omega; B) = 0$$

$$\text{and} \quad \frac{\partial F(\omega, B)}{\partial B}(V) \quad = \quad -\omega^2 \left[\psi(a_2) + \frac{i}{z\nu_2} \partial_x \psi(a_2) \right] \int_{a_1}^{a_2} \theta^2(s, \omega; B) \ V(s) \ \mathrm{d}s.$$

To obtain (4.3), it remains to note that

$$\psi(a_2) + \frac{i}{z\nu_2} \partial_x \psi(a_2) = -\frac{1}{\partial_x \theta(a_2)}.$$

The last identity easily follows from the constancy of the Wronskian $\begin{vmatrix} \theta & \psi \\ \partial_x \theta & \partial_x \psi \end{vmatrix} \equiv 1$.

Differentiating (3.2) and (A.5) w.r.t. z, we see that $\partial_z \theta$ is given by the solution y of the problem

$$y''(x) + z^2 B(x) y(x) = f(x), \quad y(a_1) = 0, \ y'(a_1) = -i\nu_1$$

with $f(x) = -2zB(x)\theta(x, z; B)$ and that $\partial_z \partial_x \theta = \partial_x \partial_z \theta$. Hence, for $z = \omega \in \Sigma(B)$,

$$\partial_z F(\omega; B) = \partial_z \theta(a_2, \omega; B) + \frac{i\partial_z \partial_x \theta(a_2, \omega; B)}{\omega\nu_2} - \frac{i\partial_x \theta(a_2, \omega; B)}{\omega^2\nu_2} =$$

$$= y(a_2) + \frac{iy'(a_2)}{\omega\nu_2} - \frac{i\partial_x \theta(a_2, \omega; B)}{\omega^2\nu_2}.$$

Solving for y by variation of parameters as before, one can easily get (4.2).

A.3 Admissible high frequencies, the proof of Proposition 6.5 (iii)

It is enough to prove the statement of Proposition 6.5 (iii) under the assumption that $b_{1,2}$ are constants satisfying $b_1 < b_2$. We consider in details several essentially different cases of arrangement of the intervals $[\nu_1, \nu_2]$ and $[b_1^{1/2}, b_2^{1/2}]$, and very briefly the other cases (for which the proof can be obtained by simple modification of arguments).

Step 1. For the cases when $0 < b_1^{1/2} < b_2^{1/2} < \nu_1$ or $\nu_2 \leq b_1^{1/2}$, let us show that

$$\mathrm{Re}\,\Sigma[\mathbb{A}] \supset \left\{ |\alpha| \geq \frac{\pi}{(a_2 - a_1) b_2^{1/2}} \left[\frac{b_1^{1/2}}{b_2^{1/2} - b_1^{1/2}} \right] \right\}. \tag{A.9}$$

Indeed, for each $n \in \mathbb{N}$ consider the intervals $\mathcal{S}_n$ covered by $\mathrm{Re}\,\omega_n(b)$ when a constant b changes in the range $(b_1, b_2]$. By (3.7), $\mathcal{S}_n = \frac{\pi n}{a_2 - a_1} [b_2^{-1/2}, b_1^{-1/2})$. If the right end of $\mathcal{S}_n$ is $\geq$ than the left end of $\mathcal{S}_{n+1}$ for all $n \geq n_0 \in \mathbb{N}$, then $\mathrm{Re}\,\Sigma[\mathbb{A}]$ contains the set $\cup_{n \geq n_0} \mathcal{S}_n = \left[\frac{\pi n_0}{(a_2 - a_1) b_2^{1/2}}, +\infty \right)$ and also its reflection w.r.t. 0, see Lemma 3.3 (ii). Estimating n_0 from the inequality $b_1^{-1/2} n \geq b_2^{-1/2}(n+1)$, one obtains (A.9).

Step 2. Slightly modifying arguments of Step 1, one can prove that:

$$\operatorname{Re}\Sigma[\mathbb{A}] \supset \left\{ |\alpha| \geq \frac{\pi}{(a_2 - a_1)b_2^{1/2}}\left(\frac{1}{2} + \left\lceil \frac{b_1^{1/2}}{b_2^{1/2} - b_1^{1/2}} - \frac{1}{2} \right\rceil\right) \right\} \quad \text{if } \nu_1 \leq b_1^{1/2} < b_2^{1/2} < \nu_2; \quad \text{(A.10)}$$

$$\operatorname{Re}\Sigma[\mathbb{A}] \supset \left\{ |\alpha| > \frac{\pi}{(a_2 - a_1)b_2^{1/2}}\left(\frac{1}{2} + \left\lceil \frac{b_1^{1/2}}{b_2^{1/2} - b_1^{1/2}} + \frac{1}{2} \right\rceil\right) \right\} \quad \text{if } \nu_1 = b_1^{1/2} < b_2^{1/2} = \nu_2; \quad \text{(A.11)}$$

$$\operatorname{Re}\Sigma[\mathbb{A}] \supset \left\{ |\alpha| > \frac{\pi}{(a_2 - a_1)b_2^{1/2}} \right\} \quad \text{if } 0 = b_1^{1/2} < b_2^{1/2} \leq \nu_1; \quad \text{(A.12)}$$

and that,

in the cases $0 < b_1 < b_2^{1/2} = \nu_1$ and $\nu_1 < b_1^{1/2} < b_2^{1/2} = \nu_2$

(A.9) and (A.10), resp., are valid after the change of $|\alpha| \geq \ldots$ to $|\alpha| > \ldots$. (A.13)

Step 3. Let us show that in the case $\nu_1 \leq b_1^{1/2} < \nu_2 < b_2^{1/2}$,

$$\operatorname{Re}\Sigma[\mathbb{A}] \supset \left\{ |\alpha| \geq \frac{\pi}{(a_2 - a_1)b_2^{1/2}}\left\lceil \frac{\frac{1}{2}b_2^{1/2} + b_1^{1/2}}{b_2^{1/2} - b_1^{1/2}} \right\rceil \right\}. \quad \text{(A.14)}$$

Now, the sets $\mathcal{S}_n$ covered by $\operatorname{Re}\omega_n(b)$ have gaps. Indeed, assume that $b_1 > 0$. Then

$$\mathcal{S}_n = \frac{\pi}{(a_2 - a_1)}\left[\frac{n}{b_2^{1/2}}, \frac{n}{\nu_2}\right) \cup \frac{\pi}{(a_2 - a_1)}\left(\frac{n + 1/2}{\nu_2}, \frac{n + 1/2}{b_1^{1/2}}\right).$$

The gap is closed by the sets $\mathcal{S}_{n-1}$ and $\mathcal{S}_{n+1}$ as soon as $(n - 1/2)b_1^{-1/2} \geq (n + 1)b_2^{-1/2}$. The latter is valid if $n \geq \frac{\frac{1}{2}b_2^{1/2} + b_1^{1/2}}{b_2^{1/2} - b_1^{1/2}}$. This proves (A.14) for $b_1 \neq 0$. To prove (A.14) in the case $0 = b_1^{1/2}$, it is enough to take the limit $b_1 \to 0$.

Step 4. Modifying arguments of Step 3, it can be shown that

$$\operatorname{Re}\Sigma[\mathbb{A}] \supset \left\{ |\alpha| > \frac{\pi}{(a_2 - a_1)b_2^{1/2}}\left(\frac{1}{2} + \left\lceil \frac{\frac{3}{2}b_1^{1/2}}{b_2^{1/2} - b_1^{1/2}} \right\rceil\right) \right\} \quad \text{if } 0 < b_1^{1/2} < \nu_1 < b_2^{1/2} \leq \nu_2; \text{(A.15)}$$

$$\operatorname{Re}\Sigma[\mathbb{A}] \supset \left\{ |\alpha| \geq \frac{\pi}{(a_2 - a_1)b_2^{1/2}}\left\lceil \frac{b_2^{1/2} + b_1^{1/2}}{b_2^{1/2} - b_1^{1/2}} \right\rceil \right\} \quad \text{if } 0 < b_1^{1/2} < \nu_1 \leq \nu_2 < b_2^{1/2}; \text{(A.16)}$$

$$\operatorname{Re}\Sigma[\mathbb{A}] \supset \left\{ |\alpha| > \frac{3}{2}\frac{\pi}{(a_2 - a_1)b_2^{1/2}} \right\} \quad \text{if } 0 = b_1^{1/2} < \nu_1 < b_2^{1/2} \leq \nu_2; \text{(A.17)}$$

$$\operatorname{Re}\Sigma[\mathbb{A}] \supset \left\{ |\alpha| \geq \frac{2\pi}{(a_2 - a_1)b_2^{1/2}} \right\} \quad \text{if } 0 = b_1^{1/2} < \nu_1 \leq \nu_2 < b_2^{1/2}. \text{(A.18)}$$

Thus, (6.2) follows from the above inclusions.

A.4   Solvability of bang-bang equations, the proof of Theorem 10.4

It is enough to consider the case $z^2 \in \overline{\mathbb{C}_-}$ (see the explanations to (5.2) in Section 5).

Lemma A.1. *Let (10.3) be fulfilled. Let $z^2 \in \overline{\mathbb{C}_-}$. Then there exists $\varepsilon = \varepsilon(a) > 0$ such that problem (10.5), (10.4) has a unique solution y on $(a - \varepsilon, a + \varepsilon) \cap [a_1, a_2]$.*

Proof. Firstly, if $b_1 = b_2$ in a neighborhood of a, the statement is obvious.

Next, to be specific, assume that $a < a_2$ and that for any $\delta > 0$ the measure of the set of $x \in (a, a + \delta)$ such that $b_1(x) \neq b_2(x)$ is positive. Then (10.3) yields

$$\text{for any } \delta > 0, \qquad \text{meas}\{x \in (a, a + \delta) \ : \ b_j(x) > 0\} > 0, \qquad j = 1, 2. \tag{A.19}$$

It is enough to prove the lemma for the interval $[a, a + \varepsilon)$ with small enough $\varepsilon > 0$.

Uniqueness. Assume that there exists at least one solution y to (10.4), (10.5). Our goal is to prove that there exist $\varepsilon > 0$ and $B_0(\cdot)$ such that for any solution y to (10.4), (10.5) the function $\mathcal{B}(y)(\cdot)$ coincides (in L^1-sense) with $B_0(\cdot)$ on $(a, a + \varepsilon)$. This and the structure of equation (10.5) immediately implies uniqueness.

Case 1. If $c_0 = c_1 = 0$, any solution y to (5.1), (10.4) is trivial and $\mathcal{B}(y) \equiv b_1$.

Case 2. If $c_0^2 \notin \mathbb{R}$ or $\partial_x y^2(a) = 2c_0 c_1 \notin \mathbb{R}$, then, for a certain nonempty interval $(a, a + \varepsilon)$, the sign of $\operatorname{Im} y^2(x)$ is completely determined by $c_0 and c_1$. This easily gives the desired statement with B_0 equal to one of the constraints (either b_1, or b_2) on $(a, a + \varepsilon)$.

Case 3. The case when $c_0 = 0$ and $c_1^2 \notin \mathbb{R}$ is analogous to the Case 2.

Case 4. Assume that $c_0^2 \in \mathbb{R} \setminus \{0\}$, $c_0 c_1 \in \mathbb{R}$, and $z^2 \in \mathbb{R}$. Then either $c_0, c_1 \in \mathbb{R}$, or $c_0, c_1 \in i\mathbb{R}$. Moreover, any solution y stays on the same line ($\mathbb{R}$ or $i\mathbb{R}$) since $z^2 \in \mathbb{R}$. Therefore $y^2(x) \in \mathbb{R}$ and $\mathcal{B}(y) = b_1$.

Case 5. The case when $c_0 = 0$, $c_1^2 \in \mathbb{R} \setminus \{0\}$, and $z^2 \in \mathbb{R}$ is analogous to Case 4.

Case 6. Assume that $c_0^2 \in \mathbb{R} \setminus \{0\}$, $c_0 c_1 \in \mathbb{R}$, and $z^2 \in \mathbb{C}_-$. Consider an arbitrary continuous in a vicinity of a branch $\arg_* y(x)$ of $\arg y$. Since $\overline{c_0} c_1 \in \mathbb{R}$, the arguments of the proof of Lemma 5.2 (i), (iv) imply

$$\partial_x \arg_* y = \frac{\operatorname{Im}(\overline{y} \partial_x y)}{|y|^2} \text{ and } \operatorname{Im}[\overline{y(x)} \partial_x y(x)] = -\operatorname{Im} z^2 \int_a^x |y(s)|^2 \mathcal{B}(y)(s) \mathrm{d}s.$$

It follows from these equalities and (A.19) that $\partial_x \arg_* y(x) > 0$ on $(a, a + \varepsilon_1(y))$ with a certain $\varepsilon_1(y) > 0$. So for any solution y,

$$\mathcal{B}(y)(x) = \begin{cases} b_2(x), & \text{if } c_0^2 > 0 \\ b_1(x), & \text{if } c_0^2 < 0 \end{cases} \qquad \text{on } (a, a + \varepsilon_1(y)). \tag{A.20}$$

This easily implies that it is possible to choose $\varepsilon_1(y)$ in (A.20) independently of a solution y.

Case 7. Assume that $c_0 = 0$, $c_1 \in \mathbb{R} \setminus \{0\}$, and $z^2 \in \mathbb{C}_-$. Since

$$y'(x) = c_1 - z^2 \int_a^x y(s) \mathcal{B}(y)(s) \mathrm{d}s = c_1 - z^2 \int_a^x \int_a^s \mathcal{B}(y)(s) y'(t) \mathrm{d}t \mathrm{d}s, \tag{A.21}$$

we see that

$$\operatorname{Im} y'(x) = -\int_a^x \int_a^s \mathcal{B}(y)(s) \operatorname{Im}[z^2 y'(t)] \mathrm{d}t \mathrm{d}s. \tag{A.22}$$

Hence, for x slightly greater than a, the complex number $y'(x)$ lies in the same open quadrant as $c_1 - i \operatorname{Im}[z^2 c_1]$. Due to the equality $y(x) = \int_a^s y'(s) \mathrm{d}s$, the numbers $y(x)$ lie in the same open quadrant. So the sign of c_1 completely determines $\mathcal{B}(y)(x)$ on $(a, a + \varepsilon_1(y))$ with a certain $\varepsilon_1(y) > 0$. As before, it is easy to see that $\varepsilon_1(y)$ can be chosen independently of a solution y.

Case 8. When $c_0 = 0$, $c_1 \in i\mathbb{R} \setminus \{0\}$, and $z^2 \in \mathbb{C}_-$, one can use arguments similar to Case 7 with the change of (A.22) to

$$\operatorname{Re} y'(x) = -\int_a^x \int_a^s \mathcal{B}(y)(s) \operatorname{Re}[z^2 y'(t)] \mathrm{d}t \mathrm{d}s.$$

This completes the proof of uniqueness.

Existence. The above arguments also provide an algorithm for assigning $B_0(x)$ the values of $b_1(x)$, or $b_2(x)$ for $x > a$ such that there exists a solution y to $y'' = -z^2 B_0 y$ on a certain interval $(a, a + \varepsilon)$ satisfying the initial conditions (10.4) and additionally $B_0(x) = \mathcal{B}(y)(x)$ on $(a, a + \varepsilon)$. Then y is a solution to (5.1), (10.4) on $(a, a + \varepsilon)$. This completes the proof. $\quad\square$

Assume that (5.1), (10.4) has a unique solution y on (x_1, x_2) with $x_1 < a < x_2 < +\infty$.

Lemma A.2. *There exist* $\lim_{x \to x_2-} y(x)$ *and* $\lim_{x \to x_2-} y'(x)$.

Proof. The statement follows from the Bellman-Gronwall lemma, from the integral reformulation of problem (10.5), (10.4) given by the equation

$$y(x) \;=\; c_0 + c_1(x - a) - z^2 \int_a^x (x - s)\, y(s) \mathcal{B}(y)(s) \mathrm{d}s, \qquad (A.23)$$

and from the fact that $|\mathcal{B}(y)(\cdot)| \leq b_2(\cdot) \in L^1_{\mathbb{R}}(x_1, x_2)$ for arbitrary y. $\quad\square$

This lemma, the equality $y'(x) = c_1 - z^2 \int_a^x y(s) \mathcal{B}(y)(s) \mathrm{d}s$, and (A.23) imply that y and y' can be considered as absolutely continuous functions on $[a, x_2]$. This arguments allow us to extend y on $[a, a_2]$. The resulting solution is unique due to Lemma A.1. This completes the proof of Theorem 10.4.

Acknowledgements

References

[1] Y. Akahane, T. Asano, B. Song, and S. Noda, High-Q photonic nanocavity in a two-dimensional photonic crystal, *Nature* **425** (2003), 944–947.

[2] I. Alexandrova, Hideo Tamura, Resonance free regions in magnetic scattering by two solenoidal fields at large separation, *Journal of Functional Analysis* **260** (2011), 1836–1885.

[3] S. Burger, J. Pomplun, F. Schmidt, L. Zschiedrich. Finite-element method simulations of high-Q nanocavities with 1D photonic bandgap, *Proc. SPIE* Vol. 7933 (2011), 79330T (Physics and Simulation of Optoelectronic Devices XIX).

[4] J.V. Burke, A.S. Lewis, M.L. Overton, Optimal stability and eigenvalue multiplicity, *Foundations of Computational Mathematics* **1(2)** (2001), 205–225.

[5] S.J. Cox, M.L. Overton, Perturbing the critically damped wave equation, *SIAM Journal on Applied Mathematics* **56 (5)** (1996), 1353–1362.

[6] S. Cox, E. Zuazua, The rate at which energy decays in a damped string, *Communications in Partial Differential Equations* **19** (1994), no. 1-2, 213–243.

[7] S. Cox, E. Zuazua, The rate at which energy decays in a string damped at one end, *Indiana Univ. Math. J.* **44** (1995), no.2, 545–573.

[8] J. Dieudonné, Foundations of modern analysis. Academic Press, New York-London, 1969.

[9] D.C. Dobson, F. Santosa, S.P. Shipman, M.I. Weinstein, Resonances of a potential well with a thick barrier, *SIAM Journal on Applied Mathematics* **73(4)** (2013), 1489–1512.

[10] R. Froese, Asymptotic distribution of resonances in one dimension, *J. Differential Equations* **137** (1997), no.2, 251–272.

[11] Yiqi Gu, Xiaoliang Cheng, A Numerical Approach for Defect Modes Localization in an Inhomogeneous Medium, *SIAM Journal on Applied Mathematics* **73(6)** (2013), 2188–2202.

[12] G.M. Gubreev, V.N. Pivovarchik, Spectral analysis of the Regge problem with parameters, *Funct. Anal. Appl.* **31** (1997), no. 1, 54–57.

[13] E.M. Harrell, General lower bounds for resonances in one dimension, *Communications in Mathematical Physics* **86(2)** (1982), 221–225.

[14] E.M. Harrell, R. Svirsky, Potentials producing maximally sharp resonances, *Transactions of the American Mathematical Society* **293(2)** (1986), 723–736.

[15] P. Heider, D. Berebichez, R.V. Kohn, and M.I. Weinstein, Optimization of scattering resonances, *Struct. Multidisc. Optim.* **36** (2008), 443–456.

[16] J.W. Jeong, S.C. Shin, I.L. Lyubchanskii, V.N. Varyukhin, Strain-induced three-photon effects, *Physical Review B* **62(20)** (2000), 13455–13463.

[17] J.D. Joannopoulos, S.G. Johnson, J.N. Winn, R.D. Meade, Photonic Crystals: Molding the Flow of Light. Princeton University Press, 2008.

[18] I.S. Kac, M.G. Krein, On the spectral functions of the string, Supplement II in F. Atkinson, *Discrete and continuous boundary problems*, Mir, Moscow 1968; Engl. transl.: *Amer. Math. Soc. Transl. (2)* **103** (1974), 19–102.

[19] C.-Y. Kao, F. Santosa, Maximization of the quality factor of an optical resonator, *Wave Motion* **45** (2008), 412–427.

[20] I.M. Karabash, Optimization of quasi-normal eigenvalues for Krein-Nudelman strings, *Integral Equations and Operator Theory* **75** (2013), no.2, 235–247.

[21] I.M. Karabash, Optimization of quasi-normal eigenvalues for 1-D wave equations in inhomogeneous media; description of optimal structures, *Asymptotic Analysis* **81** (2013) no.3-4, 273–295.

[22] I.M. Karabash, Nonlinear eigenvalue problem for optimal resonances in optical cavities, *Math. Model. Nat. Phenom.* **8** (2013), no.1, pp. 143–155.

[23] I.M. Karabash, Pareto optimal structures producing resonances of minimal decay under L^1-type constraints, *Journal of Differential Equations* **257** (2014), no.2, 374–414.

[24] M.G. Krein , A.A. Nudelman, On direct and inverse problems for the boundary dissipation frequencies of a nonuniform string, *Dokl. Akad. Nauk SSSR* **247** (1979), no. 5, 1046–1049 (Russian); Engl. transl.: *Soviet Math. Dokl.* **20** (1979), no.4, 838–841.

[25] M.G. Krein, A.A. Nudelman, Some spectral properties of a nonhomogeneous string with a dissipative boundary condition, *J. Operator Theory* **22** (1989), 369–395 (Russian).

[26] E. Kuramochi, H. Taniyama, T. Tanabe, K. Kawasaki, Y.G. Roh, M. Notomi, Ultrahigh-Q one-dimensional photonic crystal nanocavities with modulated mode-gap barriers on SiO_2 claddings and on air claddings, *Optics express* **18(15)** (2010), 15859–15869.

[27] R. Lang, M.O. Scully, W.E. Lamb, Why is the Laser Line So Narrow? A Theory of Single-Quasimode Laser Operation, *Phys. Rev. A* **7** (1973), no.5, 1788–1797.

[28] P.D. Lax, R.S. Phillips, A logrithmic bound on the location of the poles of the scattering matrix, *Archive for Rational Mechanics and Analysis* **40(4)** (1971), 268–280.

[29] X. Liang, S.G. Johnson, Formulation for scalable optimization of microcavities via the frequency-averaged local density of states, *Optics express* **21(25)** (2013), 30812–30841.

[30] A. Maassen van den Brink, K. Young, Jordan blocks and generalized bi-orthogonal bases: realizations in open wave systems, *J. Phys. A* **34** (2001), no.12, 2607–2624.

[31] B. Maes, J. Petráček, S. Burger, P. Kwiecien, J. Luksch, I. Richter, Simulations of high-Q optical nanocavities with a gradual 1D bandgap, *Optics express* **21(6)** (2013), 6794–6806.

[32] M. Maksimovic, M. Hammer, E. van Groesen, Field representation for optical defect resonances in multilayer microcavities using quasi-normal modes, *Optics Communications* **281(6)** (2008), 1401–1411.

[33] M. Notomi, E. Kuramochi, H. Taniyama, Ultrahigh-Q nanocavity with 1D photonic gap, *Optics Express* **16(15)** (2008), 11095–11102.

[34] F. Pfeiffer, C. David, M. Burghammer, C. Riekel, T. Salditt, Two-dimensional x-ray waveguides and point sources, *Science* **297(5579)** (2002), 230–234.

[35] F. Pfeiffer, U. Mennicke, T. Salditt, Waveguide-enhanced scattering from thin biomolecular films, *Journal of applied crystallography* **35(2)** (2002), 163–167.

[36] V. Pivovarchik, C. van der Mee, The inverse generalized Regge problem, *Inverse Problems* **17** (2001), no.6, 1831–1845.

[37] J. Pöschel, E. Trubowitz, Inverse spectral theory. Pure and Applied Mathematics 130, Academic Press, Boston, 1987.

[38] M. Reed, B. Simon, Methods of modern mathematical physics. IV. Analysis of operators. Academic Press, New York-London, 1978.

[39] R. T. Rockafellar, Convex Analysis. Princeton University Press, 1970.

[40] F. Schenk, Optimization of resonances for multilayer x-ray resonators. Universitätsverlag Göttingen, 2011.

[41] G. Schmidt, Spectral and scattering theory for Maxwell's equations in an exterior domain, *Archive for Rational Mechanics and Analysis* **28(4)** (1968), 284–322.

[42] A. Settimi, S. Severini, B.J. Hoenders, Quasi-normal-modes description of transmission properties for photonic bandgap structures, *J. Opt. Soc. Am. B* **26** (2009), 876–891.

[43] K. Ujihara, Quantum theory of a one-dimensional optical cavity with output coupling. Field quantization, *Physical Review A* **12(1)** (1975), 148–158.

[44] M.M. Vainberg, V.A. Trenogin, Theory of branching of solutions of non-linear equations. Noordhoff, 1974.

[45] J. Vučković, M. Lončar, H. Mabuchi, A. Scherer, Optimization of the Q factor in photonic crystal microcavities, *IEEE Journal of Quantum Electronics* **38(7)** (2002), 850–856.

[46] L.A. Woldering, A.P. Mosk, R.W. Tjerkstra, W.L. Vos, The influence of fabrication deviations on the photonic band gap of three-dimensional inverse woodpile nanostructures, *Journal of Applied Physics* **105(9)** (2009), 093108, 10 p.